Fall 1987 – Junior – 1 – Questions

F87J1

How many natural numbers n are there such that $\frac{2}{5} < \frac{n}{17} < \frac{11}{13}$?

F87J2

An interior diagonal of a polyhedron is a line segment connecting two if the polyhedron's vertices, but not lying entirely on a face of the polyhedron. How many interior diagonals does a cube have?

F87J3

Find the smallest positive integer which has exactly five distinct (positive) prime divisors.

F87J4

Points A and B are on side OQ of right angle PQR. A circle passing through points A and B is tangent to ray OP. If OA = 8 and OB = 12, find the length of the radius of this circle.

F87SJ5

For how many integer values of n is $n^5 + 3$ evenly divisible by $n^2 + 1$?

F87J6

In a rectangular coordinate system, points M(3,8) and P(11,4) are midpoints of opposite sides of a square. Find the largest ordinate (y-coordinate) belonging to a vertex of this square.

Answers

1. 8
2. 4
3. 2310
4. 10
5. 5
6. 12

Fall 1987 – Junior – 2 – Questions

F87J7

Find the smallest positive integer which has exactly five distinct positive (not necessarily prime) divisors.

F87J8

In triangle ABC, $m\angle C = 90$, $m\angle A = 30$ and $BC = 2$. A circle with its center at A divides the triangle into two regions of equal area. Find the <u>area</u> of this entire circle.

F87J9

Huey, Dewey and Louie played cards. Each started with the same amount of money. At the end of the evening, the ratio of the amounts of money they have (in the order mentioned above) is 3:4:5. By what percentage did Louie increase his original amount of money?

F87J10

In triangle ABC, AB = 5, BC = 6, and CA = 7. The bisector of angle B divides the bisector of angle A into two segments. Find the ratio of the larger of these segments to the smaller.

F87J11

The roots of the equation $2x^2 - 5x + 3 = 0$ are the reciprocals (multiplicative inverses) of the roots of the equation $x^2 + ax + b = 0$. Find the ordered pair (a,b) of real numbers.

F87J12

In trapezoid ABCD, the bisector of angle A passes through the midpoint M of leg CD. If AB = 3 and AM= $\sqrt{5}$, compute the length of MB.

Answers

7. 16
8. $12\sqrt{3}$
9. 25 or 25%
10. 2 or 2:1
11. (-5/3, 2/3): ordered pair required
12. 2

Fall 1987 – Junior – 3 – Questions

F87J13

Three people had 20 pennies each. They played a game of cards. At the end of the game, each person had a different number of pennies, and the person with the most pennies had five times the number of pennies as the person with the fewest pennies. At most, how many pennies did the person with the most pennies have?

F87J14

The roots of e equation $x^2 + ax + b = 0$ are each three more than the roots of the equation $x^2 - 7x - 7 = 0$. Find the ordered pair (a,b) of real numbers.

F87J15

Find the natural number N such that $91 \le N \le 100$ and N has the largest number of positive (not necessarily prime) divisors.

F87J16

In triangle ABC, the medians to sides AC and BC are perpendicular. If AC = c, AC = b, and BC = a, find the ratio $\left(a^2 + b^2\right) : c^2$.

F87J17

In a rectangular coordinate system, the coordinates of point A are (0,0), those of point B are (3,4) and those of point C are (10,0). Find the slope of the line along which the bisector of angle A lies.

F87J18

Find all real numbers x such that $\sqrt[3]{2-x} + \sqrt{x-1} = 1$.

Answers

13. 40 or 40 cents
14. (-13,23): ordered pair required
15. 96
16. 5:1 or 5
17. 1/2
18. 1, 2, 10: all three solutions required

Fall 1987 – Junior – 1 – Solutions

F87J1

If 2/5 < n/17, then n > 34/5 = 6.8, so $n \geq 7$. If 11/13 > n/17, then n < 187/13 < 15, so $n \leq 14$. Hence $7 \leq n \leq 14$, and there are eight such natural numbers.

F87J2

A cube has 8 vertices. From each of these, only one vertex cannot be reached by traveling along an edge or a face. Hence, each vertex corresponds to one interior diagonal. But since a diagonal has two endpoints, each interior diagonal corresponds to two vertices, and number of interior diagonals must be half the number of vertices, or four.

F87J3

The number described is clearly the product of the five smallest distinct (positive) primes. This number is $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$.

F87J4

If X is the center of the circle described, then the radius of the circle is equal To OY (see diagram). But in rectangle XPOY, PX = OY. Since centerline XY is perpendicular to Chord AB, Y is the midpoint of chord AB, so OY = (OA + OB)/2 = 10.

F87SJ5

By long division, $n^2+1 \overline{\smash{)}\, n^5+0n^4+0n^3+0n^2+0n+3}$ with quotient n^3-n, R. n + 3

So $n^5+3 = (n^3-n)(n^2+1)+(n+3)$, and either n + 3 = 0 and n = 3, or (as a necessary condition) $n^2+1 \leq n+3$, or $n^2-n-2 = (n+1)(n-2) \leq 0$, which happens only if n = -1, 0, 1, 2 (four more values). Checking these possible values, find that all five work.

F87J6

The midpoint of line segment MP has coordinates (7,6). The two shaded triangles (see diagram) are congruent, so if … is the midpoints of side AD, then the coordinates of Q are (9,10). Then AQXM is a parallelogram, so that the sums of the coordinates of opposite vertices are equal. Since the sum of the ordinates of points M and Q is 18, this is also the sum of the ordinates of A and X. Thus point A has an ordinate of 12.

Fall 1987 – Junior – 2 – Solutions

F87J7

If the required number is N, then its divisors are made up of products of combinations of its (not necessarily distinct) prime divisors. Hence we must find the smallest number with five combinations of prime divisors. It is not hard to see that this number is $2^4 = 16$.
In general, if $N = p^a q^b r^c \ldots$, where p, q, r, … are the distinct prime factors of N, then the number of divisors of N is (a + 1)(b + 1)(c + 1)… This is not really a problem in number theory, but rather in combinatorics.

F87J8

The area of the given triangle is $(1/2)(2)(2\sqrt{3}) = 2\sqrt{3}$ (see diagram), so the area of that portion of the triangle contained within the circle is $\sqrt{3}$. Since 360/30 = 12, this is 1/12 of the entire circle, whose area is then $12\sqrt{3}$.

F87J9

At the end of the evening, the amounts the three players have can be represented by 3x, 4x, and 5x, for some real number x. Then the total of all their money is 12x, and they must have each started with 4x. Hence Louie increased his amount from 4x to 5x, an increase of 25%.

F87J10

We present two solutions. Each uses the theorem that the angle bisector of a triangle divides the side to which it is drawn in the ratio of the other two sides of the triangle.
Solution I: Using mass points, assign a weight of 7 to B, of 5 to C, and of 6 to A. Since BT:TC = 5:7 (see diagram), 7B + 5C = 12T. Similarly, 6A + 5C = 11U, and the system balances at X (the intersection of the angle bisectors) with weight 18. Hence AX:XT = 12:6 = 2:1.
Solution II: Since BT:TC = 5:7, we can write BT = 5q, TC = 7q, and 5q + 7q = 6 = BC, so q = 1/2, and BT = 5/2.
Now in triangle ABT, BX is an angle bisector, so AX:XT = AB:BT = 5:(5/2) = 2:1.
In general, if the lengths of the sides of a triangle are a, b, and c, the bisector of the angle between sides of lengths a and b is divided in the ratio (a + b):c.
How many ways can you think of to establish this fact?

F87J11

One could solve the equation, then form a new equation with the required roots.
A more general technique would be to use the given equation to describe its roots. That is, let $y = 1/x$, and ask: if $2x^2 - 5x + 3 = 0$, then what must be true about y?
The answer is that $x = 1/y$, so that $2/y^2 - 5/y + 3 = 0$, or $3y^2 - 5y + 2 = 0$. Since the lead coefficient of the required equation must be 1, we must write $y^2 - 5y/3 + 2/3 = 0$, and (a,b) = (-5/3, 2/3).

For generalizations of this technique, see the section on “transformation of equations” in any book on the theory of equations.

F87J12

Drawing $MN \| BC$, we find that $\angle NAM = \angle DAM = \angle AMN$, so that triangle ANM is isosceles, and AN = NM. Since median MN in triangle ABM is half of side AB, it follows that triangle AMB is right angled with hypotenuse AB. Hence $BM^2 = AB^2 - AM^2 = 9 - 5 = 4$, and $BM = 2$.

Fall 1987 – Junior – 3 – Solutions

F87J13

If the poorest person had x pennies at the end, then the richest had 5x, and the middle person had x + a, for some positive integer a. Since there are 60 pennies altogether, then x + x+a + 5x = 60, or 7x + a = 60. For 5x to be large, x itself must be large, so 7x must be large. The largest 7x can be 56, so x = 8, and 5x = 40.

F87J14

Direct solution of the given equation id tedious. An extension of the technique of F87J11 is much quicker.

Let $y = x + 3$, so that $x = y - 3$. If $x^2 - 7x - 7 = 0$, then $(y-3)^2 - 7(y-3) - 7 = 0$, or $y^2 - 6y + 9 - 7y + 21 - 7 = 0$, or $y^2 - 13y + 23 = 0$.

F87J15

See the solution to F87J7. The number of divisors a natural number has depends on the number of its (not necessarily distinct) prime factors. Of the numbers in the given range, 97 is prime, and 91, 93, 94, and 95 are the products of two distinct primes. These clearly do not have many divisors.

Of the other numbers, 92, 98, and 99 are of the form p^2q, for distinct primes p and q, so it has 12 divisors. This is the largest number of divisors in the given range.

F87J16

Suppose the medians meet at point M (see diagram). If $MP = x$, $MQ = y$, then $MB = 2y$, $MA = 2x$ (since the medians divide each other in the ratio 1:2), and, from right triangles BMA, AMP, BMQ, we have:

$$c^2 = 4x^2 + 4y^2$$

$$a^2/4 = 4x^2 + y^2$$

$$b^2/4 = x^2 + 4y^2$$

so $(a^2 + b^2)/4 = 5x^2 + 5y^2 = 5c^2/4$, and $a^2 + b^2 = 5c^2$. Hence the required ratio is 5:1 or 5.

F87J17

Clearly $AC = 10$, and by the distance formula (or otherwise), $AB = 5$. Hence $BT{:}TC = 1{:}2$ (see diagram). Then the coordinates of T can be found from those of B and C using a weighted average (this is similar to the technique of mass points):

$T_x = (2B_x + C_x)/3 = 16/3$, $T_y = (2A_y + C_y)/2 = 8/3$.

Since AT passes through the origin, the equation of the line must be $y = x/2$, and its slope is 1/2.

F87J18

Let $a^3 = 2 - x$, $b^2 = x - 1$, so that $a + b = 1$ and $a^3 + b^2 = 1$. Substituting, we find that $a^3 + (1 - a)^2 = 1$, or $a^3 + a^2 - 2a = 0$, and $a = 0$, 1, -2. This leads to $b = 1$, 0, 3, and $x = 2$, 1, or 10.