Fall 1987 – Senior B – 1 – Questions

F87B1

The integer 1234321 is a perfect square. Find its positive square root.

F87B2

The bisectors of interior angles B and C of triangle ABC intersect at point X. If the degree-measure of angle BXC is 100, find the degree-measure of angle BAC.

F87B3

If x = -2, find the numerical value of $-\left(-x - x^{x}\right)^{-x}$.

F87B4

Find the degree-measure of n acute angle formed by two diagonals of a rectangular pentagon that do not share a common endpoint.

F87B5

The average (arithmetic mean) of n consecutive integers, starting with n, is 31. Find n.

F87B6

A train is 100 meters long. It is travelling at a constant speed of 40 kilometers per hour. F it takes 30 minutes to clear a tunnel, how many kilometers long is the tunnel?

Answers

1. 1111
2. 20
3. –49/16
4. 72 or 72°
5. 21
6. 19.9

Fall 1987 – Senior B – 2 – Questions

F87B7

Find the numerical value of $1-\frac{2}{3-\frac{4}{5-6}}$.

F87B8

Points P, Q, R, and S are chosen on sides AB, BC, CD, and DA, respectively, of square ABCD, so that AP:PB = BQ:QC = CR:RD = DS:SA = 3:1. Find the ratio of the area of square PQRS to that of ABCD.

F87B9

If $\sin x+\cos x=1/3$, find the numerical value of $\sin 2x$.

F87B10

Tom and Jerry have a reading assignment for English. Their teacher has assigned them four books to read for the month. The friends decide to save time and work by splitting the assignment. Tom will read two books, and Jerry the other two. In how many ways can they divide up the four books?

F87B11

A certain map has a scale in which 1 inch represents 20 miles. If the area of a certain county is 600o square miles, what is the area, in square inches, of the region representing this county on the map?

F87B12

The absolute value of the difference between the roots of $mx^2+5x-6=0$ is 1. Find all possible numerical values for m.

Answers

7. 5/7
8. 5:8
9. –8/9
10. 6
11. 1.5
12. 25, -1 (both required)

Fall 1987 – Senior B – 3 – Questions

F87B13

Express in simplest form the real number $\left(\sqrt{\sqrt{75}-\sqrt{12}}\right)^{-2}$.

F87B14

A famous mathematician of the 1800's once noticed that he was x years old in the year x^2. In what year was he born?

F87B15

John was five years old when his father was 41 years old. How old will John be en his father is four times older than John?

F87B16

If, for all values of x, f(x) = 2x – 7 and f(g(x)) = x, write an equation expressing g(x) explicitly in terms of x.

F87B17

Find all values of x such that: $\frac{x-2}{x-1}=\frac{x+1}{x+3}$.

F87B18

A pyramid has a square base. If each of the eight edges of the pyramid measures one unit, find the number of cubic units in the volume of the pyramid.

Answers

13. 1/3
14. 1806
15. 12
16. g(x) = (x + 7)/2 or equivalent equation
17. 5
18. $\sqrt{2}$ / 6

Fall 1987 – Senior B – 4 – Questions

F87B19

A telegram costs m cents for 12 words and s cents for each additional word. If k > 12, express in terms of m, k, and s the cost of sending a telegram of k words.

F87B20

The numerator and denominator of a fraction are both integers, and their (positive) difference is 16. If the value of the fraction lies between 5/9 and 4/7, find the fraction.

F87B21

Find all values of x for which $(x-5)^{(x^2+6x+9)} = x-5$.

F87B22

If $f(x) = -1/(1+x)$, and $g(x) = -1-x$, find the numerical value of $f(g(f(g(1987))))$.

F87B23

Huey is shorter than Dewey but taller than Louie. Louie is shorter than Huey but taller than Donald. Which is the next-to-tallest of these four characteristics?

F87B24

In trapezoid ABCD, base AD is twice as long as base BC. E is the midpoint of leg AB and F is a point on leg CD. Ray EF intersects ray AD in G, and AD = DG. Find the numerical value of the ratio CF:FD.

Answers

19. m + ks – 12s
20. 21/7
21. 5, -4, -2: all three required
22. 1987
23. Huey
24. 5:2

Fall 1987 – Senior B – 5 – Questions

F87B25

In a triangle, the degree-measure of one angle is 60 more than of another. The ratio of the lengths of the sides opposite these two angles is 2:1. Find the degree-measure of the largest angle of this triangle.

F87B26

When a certain polynomial is divided by x – 2, the remainder is 2. When the same polynomial is divided by x + 2, the remainder is –2. What is the remainder when this polynomial is divided by $x^2 - 4$?

F87B27

Find the coordinates of that point on the line 3x – 4y = 60 which is closest to the origin.

F87B28

Find all integer values of N for which $|2N - 5| < 3$.

F87B29

The sum of n terms of an arithmetic progression is 96. If the first term is n, and the last term is 2n, find the common difference of the arithmetic progression.

F87B30

Find all ordered pairs (x,y) of positive integers that satisfy $x^2 + x + 29 = y^2$.

Answers

25. 90 or $90°$
26. x
27. (36/5, 48/5)
28. 2,3 (both required)
29. 8/7
30. (28,29) or (4,7) both ordered pairs required

Fall 1987 – Senior B – 1 – Solutions

F87B1

$1234321 = 11 \cdot 112211$. It is not hard to factor this last number. In fact,

$$\begin{aligned} 112211 &= 11 \cdot 10^4 + 22 \cdot 10^2 + 11 \\ &= 11\left(10^4 + 2 \cdot 10^2 + 1\right) = 11\left(10^2 + 1\right)^2 = 11 \cdot 101^2 \end{aligned}$$

The square root, then, is $11 \cdot 101 = 1111$.

F87B2

In triangle BXC, the sum of angles XBC and XCB is 180 – 100 = 80. Since BX and CX are angle bisectors, the sum of angles ABC and ACB is twice this, or 160. This leaves 180 – 160 = 20 degrees for the measure of angle BAC.

F87B3

The given expression equals $-\left[2 - \left(-2\right)^{-2}\right]^2 = -\left[8/4 - 1/4\right]^2 = -\left(7/4\right)^2 = -49/16$.

F87B4

We can inscribe the regular pentagon in a circle. Then arc BA = arc DE = 360/5 = 72. Angle BXA is the average of these two arcs, which is also 72 degrees.

F87B5

If the average of n integers is 31, the sum of the n integers is 31n. But these integers form an arithmetic progression, whose sum can be computed as (n/2)(n + 2n – 1), since the last integer is 2n – 1. Equating these two, we find

$$31n = n^2/2 + n^2 - n/2$$
$$62n = 3n^2 - n$$
$$63n = 3n^2$$, and n = 21.

F87B6

If the length of the tunnel, in meters, is x, then the train will travel x + 100 meters before it clears the tunnel (since the engine of the train must travel the train's length once it is out of the tunnel, to pull the full length of the train out). We have:

$x + 100 = 40\left(1/2\right)\left(1000\right) = 20000$, and $x = 19900$, or 19.9 kilometers.

Fall 1987 – Senior B – 2 – Solutions

F87B7

$$1-\frac{2}{3-\frac{4}{5-6}}=1-\frac{2}{3-\frac{4}{-1}}=1-\frac{2}{3+4}$$

$$1-2/7=5/7$$

F87B8

Take AB = 4, so that AP = 3 and PB = 1. Using absolute value of area, we have $|APS|=|PBQ|=|QCR|=|RDS|=3/2$, and $|ABCD|=16$, so that $|PQRS|=16-4|APS|=16-6=10$, and $|PQRS|:|ABCD|=5:8$. Or: use the Pythagorean Theorem in triangle APS.

F87B9

Squaring the given equation, we find: $\sin^2 x+2\sin x\cos x+\cos^2 x=1/9$, or $1+\sin 2x=1/9$, and $\sin 2x=-8/9$.

F87B10

If Tom picks two of the four books to read, then Jerry must read the other two books. Hence we need only count the number of ways Tom can choose two out of four books. This is easily seen to be $4\cdot 3/2=6$ ways.

F87B11

A map is similar to the area it represents. Since the ratio of linear dimensions is 1:20, the ratio of the areas of these two similar figures is 1:400. If the map area is x, then x:600 = 1:400, and x = 1.5 square inches.

F87B12

If the roots of the given equation are p and q, with p > q, then w have: p – q = 1, p + q = -5/m, pq = -6/m.

Adding the first two equations, 2p = 1 – 5/m, and subtracting, 2q = - 1 – 5/m. Multiplying, 4pq = -1 + 25/m^2 = -24/m. Solving for m, we find: $-m^2+25=-24m$, $m^2-24m-25=0$, and $m=25$ or -1.

Fall 1987 – Senior B – 3 – Solutions

F87B13

$$\left(\sqrt[3]{\sqrt{75}-\sqrt{12}}\right)^{-2} = \left(\sqrt[3]{5\sqrt{3}-2\sqrt{3}}\right)^{-2} = \left(\sqrt[3]{3\sqrt{3}}\right)^{-2}.$$

$$= 3^{(3/2)(1/3)(-2)} = 3^{-1} = 1/3$$

F87B14

Since $42^2 = 1764$, $43^2 = 1849$, and $44^2 = 1936$, he must have been 43 in the year 1849. Thus he was born in the year 1806.
The mathematician's name was De Morgan.

F87B15

John's father is x + 36 years old whenever John s x years old. At the time referred to, x + 36 = 4x, or 3x = 36, and x = 12.

F87B16

We have 2g(x) – 7, so g(x) = (x + 7)/2.

F87B17

Cross multiplying, we find that $x^2 - 1 = x^2 + x - 6$, or $x - 6 = -1$, and $x = 5$.

F87B18

The volume of a pyramid is one-third the product of the area of the base and the height. The area of the base, here, is clearly one cubic unit. To find the height, we may look at the diagonal cross section of the figure shown below. XY is a diagonal of the base, so its length is $\sqrt{2}$. Then the length of XH is $\sqrt{2}/2$, and since XZ = 1, $ZH^2 = ZY^2 - HY^2 = 1 - 1/2 = 1/2$, so $ZH = \sqrt{2}/2$. The volume, then, is $(1/3)(1)(1)\left(\sqrt{2}/2\right) = \sqrt{2}/6$.

[diagram goes here]

Fall 1987 – Senior B – 4 – Solutions

F87B19

Cost = m + (k – 12)s = m + ks – 12s.

F87B20

If the numerator of the fraction is x, then the denominator is x + 16 (since the value of the fraction is less than 1). Since (5/9) < x/(x + 16), 5x + 80 < 9x, and x > 20.
Since x/(x + 16) < 4/7, 7x < 4x + 64, and x < 64/3 = 21 + 1/3. It follows that x = 21, and the fraction is 21/37.

F87B21

If $x-5=0$, then $x=5$, which is one solution. If $x-5$ is not zero, we can divide by it, to get $x^2+6x+9=1$, or $x^2+6x+8=(x+4)(x+2)=0$, and $x=-4$, -2. There are three solutions in all.

F87B22

We have $f(g(x)) = -1/(1+(-1-x)) = -1/(-x) = 1/x$. Hence $f(g(f(g(x)))) = f(g(1/x)) = 1/(1/x) = x$. Thus $f(g(f(g(1987)))) = 1987$.

F87B23

The first sentence implies the order Louie, Huey, Dewey (from short to tall). The second sentence implies that Donald is the shortest, so that Huey is the next-to-tallest. Note that the second sentence duplicates some information from the first.

F87B24

Draw $EX \parallel AB$. Then, if $AD = 2a$, $EX = (AD + BC)/2 = 3a/2$, and since triangles EXF, GDF are similar, $XF{:}FD = EX{:}DG = 3{:}4$, so $CF{:}FD = 10{:}4 = 5{:}2$.

Fall 1987 – Senior B – 5 – Solutions

F87B25

It is not hard to guess that the triangle is a 30-60-90 triangle. To prove this, suppose that the angles referred to are x and x + 60. Then, by the law of sines,

$$2/1 = [\sin(x+60)]/[\sin x] = [\sin x \cos 60 + \cos x \sin 60]/[\sin x]$$
$$= 1/2 + (\sqrt{3}/2)\cot x$$

Hence $\cot x = \sqrt{3}$, and $x = 30$ degrees. This makes the largest angle of the triangle 90 degrees.

F87B26

We have

(i) $P(x) = Q_1(x)(x-2) + 2$

(ii) $P(x) = Q_2(x)(x+2) - 2$.

We need to find a and b, where

(iii) $P(x) = Q_3(x)(x^2-4) + ax + b$.

From (i) and (iii), P(2) = 2 = 2a + b. From (ii) and (iii), P(-2) = -2 = -2a + b.
Solving simultaneously, we get a = 1, b = 0, so the remainder is x.

F87B27

If x = 0, y = -15, and if y = 0, x = 20. Using the Pythagorean theorem in triangle OBA (see diagram) shows that AB = 25, and computing the area of triangle OPA in two different ways, or otherwise, we find OT = 12. Then PT:OT = PT:12 = 3:5, while PO:OT = PO:12 = 4:5. Solving gives PT = 36/5 = 7.2, PO = 48/5 = 9.6. Since these two segments give the coordinates of point T, and since T is the closest point on line AB to point O, the answer is (36/5, 48/5) or equally ordered pair.

F87B28

The given condition can be written as $|N - 5/2| < 3/2$, and this can be interpreted as the set of all points which are less than 3/2 units away from 5/2. Graphically:

[graph goes here]

Since the inequality is strict, only the integers 2 and 3 satisfy the given condition.

F87B29

We have $(n/2)(n + 2n) = 96$, or $3n^2/2 = 96$, $n^2 = 64$, and $n = 8$. Then, if the common difference is d, $2n = 16 = 8 + 7d$, and $d = 7/8$.

F87B30

Completing the square gives $(x^2 + x + 1/4) + 29 = y^2 + 1/4$, or $(x + 1/2)^2 - y^2 = -115/4$, or $4y^2 - (2x+1)^2 = 115$. Factoring the difference of two squares on the left produces

$(2y+2x+1)(2y-2x-1)=115$. Since x and y are positive integers, each of the factors on the left is also an integer, and hence each must be a factor of 115. This leads to four possibilities:

$2y + 2x + 1 = 115$	23	5	1, while
$2y - 2x - 1 = 1$	5	23	115.

The last two cases may be ignored, as the first factor must be larger than the second. Adding, we have $4y = 116$ or 28, and $y = 29$ or 7. By substitution, $x = 28$ or 4, and $(x,y) = (28,29)$ or $(4,7)$.