Spring 1988 – Junior – 1 – Questions

S88J1

Compute: $\frac{11111}{22222}+\frac{33333}{44444}$.

S88J2

Peter has four sticks of lengths 2, 4,7, and 10 centimeters. He chooses three of them at random. What is the probability that the three sticks he chooses will form a triangle?

S88J3

What is the average (arithmetic mean) of the number of degrees in the angles of a nine-sided polygon?

S88J4

Ron had twenty-six matches, all of the same length. Without breaking any matches, he uses them to form the outline of a rectangle. What is the area of the largest rectangle he can form, in square match-lengths?

S88J5

If r and s are roots of the equation $x^2 - px + q = 0$ (in which p and q are real numbers), express in terms of p and q the value of $\frac{1}{r^2}+\frac{1}{s^2}$.

S88J6

In parallelogram ABCD, point X is chosen on side AB, and point Y is chosen on side AD, so that AX:XB = 1:3and AY:YD = 1:4. If diagonal AC intersects line segment XY in point P, compute the ratio AP:AC.

Answers

1. 5/4
2. 1/4
3. 140 or 140°
4. 42
5. $\left(p^2 - 2q\right)/q^2$
6. 1:9 or equivalent

Spring 1988 – Junior – 2 – Questions

S88J7

Marius had \$5 in quarters. Sulla had as many nickels as Marius had quarters. How much money did Sulla have?

S88J8

Triangle ABC is equilateral. On side AB, square ABEF is constructed, with no interior point in common with triangle ABC. Find the degree-measure of angle ECF.

S88J9

Sam the grocer uses a bicycle to supply his store. He can pedal at a rate of 30 miles per hour with no load, but can only pedal 20 miles per hour with a load of fresh fish. It takes him 48 minutes to leave his store (with no load), get a load of fresh fish at the dock, and pedal back. Assuming no time is lost in loading the fish, how many miles rfom the store is the dock?

S88J10

Find the number of square units in the area of a triangle whose sides are 4, 13, and 15 units in length.

S88J11

A discount clothing store bought a pair of pants at a price which was 34% less than the price on the ticket. The store sold the pants for a price which was 25% less than the price on the ticket. What percent of the final selling price was profit for the store?

S88J12

In triangle ABC, the median and the altitude from vertex C trisect angle ABC. Find the number of degrees in the smallest angle of triangle ABC.

Answers

7. \$1.00 (or equivalent)
8. 30 or 30°
9. 48/5 or equivalent
10. 24
11. 12 (accept also 12%)
12. 30 or 30°

Spring 1988 – Junior – 3 – Questions

S88J13

Bob had two numbers. The (positive) difference between Bob's numbers was equal to the product of his two numbers. Elizabeth took the reciprocals of each of Bob's numbers. What is the (positive) difference between Elizabeth's two numbers?

S88J14

Line segment AB is the diameter of a semicircle. Rectangle PQRS is inscribed in the semicircle so that P and Q are on diameter AB, R and S are on the semicircle, and R and S divide the semicircle into three congruent arcs. If AB = 12 cm, find number of square centimeters in area of rectangle PQRS.

S88J15

Linda chose a letter at random from the word MEET. Harold chose a letter at random from the word FEET. What is the probability that Linda and Harold will choose tha same letter?

S88J16

In right triangle ABC, leg BC = 7 and leg AC = 11. A square is constructed with hypotenuse AB as one side. The interior of the square and of triangle ABC have no points in common. Find the distance from point C to the point of intersection of the diagonals of this square.

S88J17

If x and y are positive prime integers such that $x^2 - 2y^2 = 17$, compute the numerical value of $x + y$.

S88J18

In triangle ABC, AB = 13, AC = 14, and BC = 15. A circle is inscribed in this triangle, and three tangents to this circle are drawn, each parallel to one of the triangle's sides, and diametrically opposite the circle. These three new parallels cut off three small triangles, one sharing vertex A with triangle ABC, one sharing vertex B, and one sharing vertex C. If three small circles are inscribed in these three new triangles, find the sum of the radii of these three circles.

Answers

13. 1
14. $18\sqrt{3}$
15. 5/16
16. $9\sqrt{2}$
17. 7
18. 4

Spring 1988 – Junior – 1 – Solutions

S88J1

Each numerator and each denominator is a multiple of 11111. Reducing to lowest terms, the required number is 1/2 + 3/4 = 5/4.

S88J2

Since Peter is choosing three out of four sticks, he is leaving out one stick, so he has four possible choices. For the three sticks chosen to form a triangle, the sum of the two smaller sticks must be greater than the third stick. Of the four possible choices, only {4, 7, 10} works. Hence the probability of forming a triangle is 1/4.

S88J3

Since the sum of the angles of a nonagon (nine-sided polygon) is $7 \cdot 180$, and there are nine angles, the average of the angles is (7/9)(180) = 140 degrees.

S88J4

The largest area possible would be a square, but since 26 is not a multiple of 4, he cannot form an exact square. The next best rectangle he can make is one which is closest to a square, whose length is 7 matches and whose width is 6 matches. Its area is 42 (square matches).
(It is not hard to see, with some experimentation, that Ron gains nothing by leaving matches out to form a smaller square.) In general, the largest polygon with a given perimeter—and no other constraints—will be a regular polygon.

S88J5

We have $r + s = p$, $rs = q$. The given expression can be written as
$\left(r^2 + s^2\right)/r^2s^2 = \left[\left(r + s\right)^2 - 2rs\right]/r^2s^2 = \left(p^2 - 2q\right)/q^2$.

S88J6

In working with ratios, it is often useful to introduce a set of parallel lines. To this end, we draw BM and DN parallel to XY (see diagram). Since triangles ABM, CDN are congruent, AM = NC, and we have AM:AP = AB:AX = 4, AM:AP = AD:AY = 5. Hence AC = AN + NC + AM + AN, so AC:AP + AM:AP + AN:AP = 4+5 = 9, and AP:AC = 1:9.

Spring 1988 – Junior – 2 – Solutions

S88J7

Since Sulla had 5 cents for each 25 cents of Marius, he must have 1/5 much money, or $1.00.

S88J8

Since EB = AB – BC, triangle EBC is isosceles, and $m\angle EBC = m\angle EBA + m\angle ABC = 90°+60° = 120°$. Hence $m\angle ACF = 15°$, so $m\angle ECF = m\angle BCA - m\angle BCE - m\angle FCA = 60 - 15 - 15 = 30°$.

S88J9

If the required distance is d, then d/30 + d/20 = 4/5 (since 48 minutes is 4/5 hour), or 5d/60 = 4/5, which leads to d = 48/5 miles.

S88J10

The answer is not hard to get using Hero's formula. Alternatively, we can draw the altitude to the side of 4 units (see diagram). It then becomes clear that the given triangle was chosen because it can be drawn as the "difference" between a 9-12-15 right triangle and a 5-12-13 right triangle.
In such triangle, the three sides, and altitude, and the area are all integers. These triangles are called "Heronian". Can you find others?

S88J11

If the ticket price was 7, the store bought the pants at .66T, and sold it at .75T. The difference is .09T, which is the store's profit. As a percentage of selling price, this is $100 \times .09T/.75T = (9/75)(100) = 12$ (accept also 12%).

S88J12

Since the altitude to AM in triangle AMC is also an angle bisector, this triangle is isosceles. Thus the same line (CH) is also a median, and AH = HM. If we draw $MK \perp CB$ (see diagram), triangles CMH, CMK are congruent (by SAA), so HM = MK. This means that MK = HM = AH = AM/2 = MB/2 (since AM = MB). Hence in right triangle MKB, side MK is half of hypotenuse MB, and the triangle has acute measuring $60°$ and $30°$. Thus angle B is $30°$.

It is not hard to see now that angle B is the smallest angle of triangle ABC. Indeed, since triangle CHB is a 30-60-90 triangle, angle HCB is $60°$. Hence angle ACB is $90°$, and ABC is in fact itself a 30-60-90 triangle.

Spring 1988 – Junior – 3 – Solutions

S88J13

If Bob’s numbers are x and y (with x > y), then x – y = xy. Dividing both sides of this equation by xy, we find immediately that 1/y – 1/x = 1.
Note that to get a positive difference, Elizabeth has to subtract her numbers in the “opposite order” from the way Bob subtracted his.

S88J14

If the center of the semicircle is O, then arcs AR, RS, RB are each $60°$, and angles AOS, SOR, ROB are also $60°$. Hence, in triangle ROQ, OR = 6, OQ = 3, and RQ $=3\sqrt{3}$. Since PQ = 2OQ = 6, the area of PQRS is $6 \cdot 3\sqrt{3} = 18\sqrt{3}$.

S88J15

We suppose, without loss of generality, that Linda choose first. If she chose M, which happens 1/4 of the time, Harold’s letter will definitely not match. If she chose E, which happens 1/2 the time, Harold will choose an E also half the time, so they will both have E’s 1/4 of the time. If Linda chooses the T, which happens 1/4 of the time, Harold will match her by choosing his T 1/4 of the time, so that they will both have T’s 1/16. Altogether, they will match 1/4 +1/16 = 5/16 of the time, and this is the probability that they will pick the same letter.
A tree diagram makes it easy to visualize this problem in conditional probability.

S88J16

If we “embed” the triangle and the square in the figure shown at the right, it is not hard to see that the required distance is half the sum of the legs times $\sqrt{2}$.

A similar diagram can be used to give a proof of the Pythagorean theorem