

Spring 1988 – Senior A – 1 – Questions

S88S1

Find a natural number which is equal to five times its remainder upon division by 10.

S88S2

Two circles of radius 3 and 5 are tangent externally at point A. A line is tangent to one circle at point C (distinct from A) and the other at point D. Compute the degree-measure of angle CAD.

S88S3

If $f(x) = x^2 + 5x + 4$, for all real numbers x , write an equation expressing $f(x)$ explicitly in terms of x .

S88S4

How many of the following integers are perfect squares: 16, 1156, 111556, 111155556?

S88S5

For how many real values of x is $\log_{3-x} |x| = 1$?

S88S6

If x and y are real numbers, and $x + 2y = 6$, find the smallest possible value of $x^2 + y^2$.

Answers

1. 25
2. 90 or 90°
3. $f(x) = x^2 + 3x$: equation required
4. 5
5. 3 or 3 values
6. $36/5$

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S88S7

One-third of a certain positive number is equal to five times the reciprocal (multiplicative inverse) of the number. Find the number.

S88S8

A regular hexagon (regular 6-gon) with all of its diagonals drawn has six vertices, each of which is connected by line segments to exactly five other points. Find the next largest integer n for which there exists a regular n -gon, each of whose vertices can be connected to exactly five others.

S88S9

How many positive integers divide $10!$, where the notation $10!$ means the number $10 \cdot 9 \cdot 8 \cdot 7 \cdot \dots \cdot 3 \cdot 2 \cdot 1$?

S88S10

How many ordered pairs (x,y) of integers satisfy $x^2 = y^2 + 2y + 13$?

S88S11

Compute the number $2^{\frac{1}{2} \binom{1000}{2}}$.

S88S12

In a rectangular coordinate system, two sides of a parallelogram lie along the lines $y = x - 2$ and $5y = x + 6$, and the diagonals of the parallelogram intersect at the origin. Find the length of the shorter diagonal of the parallelogram.

Answers

7. $\sqrt{15}$
8. 8
9. 270
10. 65280
11. $2\sqrt{2}$

Spring 1988 – Senior A – 3 – Questions

S88S13

The smallest perfect square integer which is a multiple of 1988 is n^2 . Find n .

S88S14

Points A, M, B, and N lie along a line in the order given, and $AM:MB = AN:NB = 2$. Compute the numerical value of the ratio $MA:AN$.

S88S15

Compute the numerical value of the expression $5^{\log_{25} b^{-3/4} c}$.

S88S16

In a rectangular coordinate system, an isosceles right triangle has its hypotenuse along the line $y = 3x + 2$, and the vertex of its right angle at $(3, -1)$. Find the length of a leg of this triangle.

S88S17

If $\tan x + \cot x = 5/2$, compute the numerical value of $\tan^4 x + \cot^4 x$.

S88S18

Gordie and his friends are on a hike, and have to cross a railroad bridge. It will take them 15 minutes to cross the bridge, but (very short) trains come along the bridge every 25 minutes. It takes a train 2 minutes to cross the bridge, in the same direction Gordie will be walking. Neglecting the length of the train, what is the probability that Gordie and his friends will cross the bridge without encountering a train?

Answers

19. 30
20. 5
21. 53
22. $2/3$
23. $\sqrt{2}$
24. (497,497) ordered pair required

Spring 1988 – Senior A – 4 – Questions

S88S19

How many integers which are perfect squares are also factors of the number 10!?

S88S20

In triangle ABC, $AB = 5$, $BC = 4$, and $CA = 3$. On the plane of triangle ABC, how many points D are there such that the set $\{A, B, C, D\}$ of four points will be symmetric with respect to some line? (A set of points is symmetric with respect to a line if each point in the set is the “mirror-image” of a point in the set, with the line as “mirror”.)

S88S21

What is the largest number of Sundays that can occur in any single calendar year?

S88S22

In square ABCD, points M, N, P and Q are the midpoints respectively of sides AB, BC, CD, and DA. Point X is the midpoint of MB, and point Y is the midpoint of PD. What fraction part of line segment XY is interior to square MNPQ?

S88S23

For each real number x such that $0 < x < \pi / 2$, a triangle is formed with one side of length $\sin x$ and another side of length $\cos x$. What is the smallest number which is larger than any possible value (the least upper bound) for the length of the third side of such a triangle?

S88S24

Find all ordered pairs (x,y) of positive integers such that $\sqrt{x} + \sqrt{y} = \sqrt{1988}$.

Answers

19. 30
20. 5
21. 53
22. $2/3$
23. $\sqrt{2}$
24. (497,497) ordered pair required

Spring 1988 – Senior A – 5 – Questions

S88S25

Find all real numbers x such that $\log_7 5 - \log_7 4 = \log_7 10 - \log_7 x$.

S88S26

When written out in decimal notation, the number $9^{\text{e}j}$ has a tens digit of t and a units digit of u . Find the ordered pair (t,u) of decimal digits.

S88S27

The number N is represented by a two-digit base ten numeral. If the two digits are reversed, and the new number is added to N , the result is a perfect square. How many such numbers N are there?

S88S28

When two fair, six-sided dice are thrown, and the sum of the faces observed, it is not hard to see that the most likely outcome is a sum of 7. When three fair, six-sided dice are thrown, and the sum of the faces observed, there are two outcomes with equally high possibilities that are the most likely outcomes. What is the sum of the faces for (either one) of these outcomes?

S88S29

A list of statements says:

- Exactly one statement on this list is false.
- Exactly two statements on this list are false.
- Exactly three statements on this list are false.
- Exactly four statements on this list are false.
- Exactly five statements on this list are false.
- Exactly six statements on this list are false.
- Exactly seven statements on this list are false.
- Exactly eight statements on this list are false.
- Exactly nine statements on this list are false.
- Exactly ten statements on this list are false.

How many statements on this list are false?

S88S30

The numbers 2, 3, 4, 5, 6, 7, 8, 9, 10 are written down. Then each pair of these numbers is multiplied together, and their product is written down. Then each set of three such numbers is multiplied together, and their product is written down. This process is continued: the product of every set of four, five, six, seven and eight numbers is written down, and finally the product of all nine is written down. The reciprocals (multiplicative inverses) of all the numbers written are then added. What is the final sum?

Answers

- 25. 8
- 26. (8,9)
- 27. 8
- 28. 10 or 11 (accept either or both)
- 29. 9
- 30. $9/2$

Spring 1988 – Senior A – 1 – Solutions

S88S1

If the number is N , then we can write $N = 10q + r = 5r$, where q is the quotient and r the remainder upon division by 10. This leads to $10q = 4r$, or $5q = 2r$. This equation implies that 5 divides r . Since $r < 10$, we must have $r = 5$ and $q = 2$. The number then is $10(2) + 5 = 25$.

S88S2

Draw AX , the common internal tangent to the two circles. Then, since tangents to a circle from a point outside are equal, $XC = XA = XD$. Thus in triangle ADC the median to one side is equal to half that side, which means that the triangle must be right-angled at A , and $m\angle CAD = 90$.

The same result can be achieved by drawing radii perpendicular to CD at points C and D , and finding pairs of supplementary angles.

S88S3

METHOD I:

Completing the square, $f(x) = x^2 + 2x + 1 + 3x + 3 = (x+1)^2 + 3x + 3$, so $f(x) = (x+1)^2 + 3x$.

METHOD II:

If $g(x) = f(x) + 4 = x^2 + 5x + 4$, then $g(x) = (x+1)^2 + 5(x-1) + 5 = x^2 + 3x$.

S88S4

It is not hard to see that in fact all numbers of the given form are perfect squares: $16 = 4^2$, $1156 = 34^2$, $111556 = 334^2$, and so on. That the pattern continues can be seen by examining the usual arithmetic multiplication algorithm.

Algebraically, if the number $111\dots155\dots56$ contains n 1's, $n-1$ 5's and a final digit 6, then it can be written as

$$\begin{aligned} & 10^n + 10^{n+1} + \dots + 10^{2n-1} + 5(10^{n-1} + 10^{n-2} + \dots + 10^2 + 10^1) + 6 \\ &= 10^n(10^n - 1)/9 + 5 \cdot 10(10^{n-1} - 1)/9 + 6 \\ &= (10^{2n} - 10^n + 5 \cdot 10^n - 50)/9 + 6 \\ &= (10^{2n} + 4 \cdot 10^n + 4)/9 \\ &= [(10^n + 2)/3]^2 \end{aligned}$$

S88S5

We must have $x < 3$, since the base of a system of logarithms cannot be negative. Then we can write $1/|x| = 3 - x$, and we need to consider $x > 0$, $x < 0$ separately.

If $x > 0$, then $1/x = 3 - x$, or $x^2 - 3x + 1 = 0$, and $x = \frac{3 \pm \sqrt{5}}{2}$, and both solutions lead to solutions for the original equation.

If $x < 0$, $1/x = x - 3$, which leads to $x^2 - 3x - 1 = 0$. The solutions of this equation are $x = \frac{3 \pm \sqrt{13}}{2}$. However, if the “plus” sign is chosen, the resulting root will be positive, hence extraneous. The other root, however, does satisfy the original equation. The three solutions can be visualized graphically as the three points of intersection of the curves $y = 1/|x|$ and $y = 3 - x$.

S88S6

The given problem can be interpreted graphically, if we think of the expression $x^2 + y^2$ as the square of the distance from the point (x,y) to the origin. This square is minimal when the distance itself is minimal. The problem further requires that the point (x,y) be on the line whose equation is $x + 2y = 6$.

Thus we must find the (perpendicular) distance from the origin to this line. The formula for this may be found in any text in analytic geometry.

Alternatively, we need the length of the altitude to the hypotenuse in right triangle OAB. We can get this, for instance, by computing the area in two ways (as half the product of the legs, and as the product of the hypotenuse and the altitude:

$$K = 3 \cdot 6 / 2 = h\sqrt{45} / 2, \text{ or } h = 18 / \sqrt{45} = 6 / \sqrt{5}. \text{ The square of this quantity is } 36/5 \text{ or } 7 \frac{1}{5}.$$

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S88S7

If the number is x , then $x/3 = 5/x$, or $x^2 = 15$, and $x = \sqrt{15}$.

S88S8

If there are n points, each connected to five others, there are $5n/2$ line segments connecting the points. Since there must be an integral number of line segments, n must be even. In fact, the next even number, which is 8, will work:

[graph goes here]

S88S9

The prime factorization of $10!$ is $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$, so the number has $9 \cdot 5 \cdot 3 \cdot 2 = 270$ factors. See any book on elementary number theory for a discussion of this function..

S88S10

Completing the square, we can rewrite the given equation as $x^2 = (y+1)^2 + 12$, or

$x^2 - (y+1)^2 = 12$. The left side factors as the difference of the two squares:

$(x - y - 1)(x + y + 1) = 12$, and we can solve this equation by considering different factorizations of 12:

$x + y + 1$	$x - y - 1$	$x + y$	$x - y$	x	y
12	1	11	2	-	-
6	2	5	3	4	1
4	3	3	4	-	-
3	4	2	5	-	-
2	6	1	7	4	-3
1	12	0	13	-	-
-12	-1	-13	0	-	-
-6	-2	-7	-1	-4	-3
-3	-4	-4	-3	-	-
-2	-6	-3	-5	-4	-1
-1	-12	-2	-11	-	-
-4	-3	-5	-2	-	-

The four possible solutions are $(4,1)$, $(4,-3)$, $(-4,-3)$, and $(-4,1)$.

S88S11

The first number in the given difference is equal to 2^{16} , while the second is 2^8 . Thus we must compute $2^{16} - 2^8 = 2^8(2^8 - 1) = 256(255) = 65280$.

S88S12

Solving the given equations simultaneously, we find that one vertex of the parallelogram is at the point with coordinates $(4,2)$. It follows that the opposite vertex has coordinates $(-4,-2)$. We can then find the other two vertices by writing their coordinates as (a,b) and $(-a,-b)$. Since one of these points is on each of the given lines, we must solve simultaneously the equations $b = a - 2$ and $-5b = -a + 6$. Adding, we find $-4b = 4$, so $b = -1$. It follows that $a = 1$, and the shorter diagonal has endpoints at $(-1,1)$ and $(1,-1)$. The length of this line segment is $2\sqrt{2}$.

Spring 1988 – Senior A – 3 – Solutions

S88S13

The prime factors of a perfect square must occur in even powers in its prime factorization. Here, since $1988 = 2^2 \cdot 7 \cdot 71$, the smallest number by which this can be multiplied to produce a perfect square is $7 \cdot 71$, or 497. The resulting perfect square is $2^2 \cdot 7^2 \cdot 71^2 = 994^2$ (which equals 988036). The answer is 994.

S88S14

If $MB = x$, then $MA = 2x$, and $AB = 3x$. Then $AN = AB + BN = 2BN$, or $3x + BN = 2BN$, so $BN = 3x$ as well, and $AN = 6x$. Then $MA:AN = 2x:6x = 1:3$.

S88S15

Since $\log_{25} a = \frac{\log_5 a}{\log_5 25}$ (for any positive number a), the given expression is equal to $\frac{\log_5 a}{5} \cdot \frac{5}{4} = \frac{\log_5 a}{4} = 5/2$.

S88S16

The distance from point C to the given line is $12/\sqrt{10}$. By symmetry, this must be the altitude to the hypotenuse of the required triangle. The length of a leg of the triangle can then be obtained by multiplying by $\sqrt{2}$.

To compute the distance from point C to the given line, we may use the method referred to in problem S88S6, or we can proceed geometrically. The line through C parallel to the given line has equation $y = 3x - 10$. If the required distance is d , then from triangle PCQ (see diagram), $CQ = 3d$ (the tangent of the angle at Q is just the slope of the given line). Since PQ is parallel to the y-axis, the length of PQ is equal to the difference in the y-intercepts of the two lines, which is 12. Using the Pythagorean theorem in triangle PCQ, we find that $d = 12/\sqrt{10}$.

S88S17

If $\tan x = a$, then $\cot x = 1/a$, and the given expression is equivalent to $a + 1/a = 5/2$.

We need to compute $a^4 + 1/a^4$. We have: $a + 1/a = 5/2$, so

$a^2 + 1/a^2 = 17/4$. Squaring once again, a similar calculation shows that

$$a^4 + 1/a^4 = 289/16 - 2 = 257/16.$$

Alternatively, we can solve directly for a . Since $5/2 = 2 + 1/2$, $a = 2$ or $a = 1/2$ (since the equation is quadratic, these are the only two roots). A direct calculation gives the result.

S88S18

The earliest “safe” time for Gordie to start walking on the bridge is just after a train comes. The latest “safe” time is when he will meet a train just at the other end of the bridge, which means he begins 13 minutes before a train comes. Thus there is an interval of 12 minutes out of every 25 during which he can start walking, without encountering a

train. Since he begins his walk at random the probability of his choosing a “safe” time is $12/25$.

A diagram, in the form of a time line, helps to visualize this situation.

Spring 1988 – Senior A – 4 – Solutions

S88S19

See problem S88S9, S88S13. The prime factorization of $10!$ is

$2^8 \cdot 3^4 \cdot 5^2 \cdot 7 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7$. In the prime factorization of a perfect square, each prime must enter an even number of times. Hence there are 5 ways to choose some factors of 2, 3 ways to choose some factors of 3, and two ways to choose some factors of 5, making $(5)(3)(2) = 30$ ways to choose a divisor of $10!$ which is also a perfect square.

S88S20

We can distinguish several cases by the position of the line of symmetry. Clearly, this line cannot contain an odd number of the points $\{A, B, C, D\}$ --otherwise the remaining points could not be paired off symmetrically. And the line cannot contain all four of the points, since even the three given points are not collinear. Thus the line must contain either two or none of the four points.

If the line contains two points, there are only three possibilities (the three sides of triangle ABC). These cases give three different positions for D . If the line contains none of the given points, then it must be the perpendicular bisector of a pair of the points. This would, in general, give three more positions for D . But triangle ABC is right-angled at C (by the converse of the Pythagorean theorem), so the perpendicular bisectors of AC and BC lead to the same point D as a fourth point: the point which is opposite point C in a rectangle.

S88S21

Certainly a normal year cannot contain more Sundays than a leap year, so we can assume without loss of generality that the year in question has 366 days. Now of any seven consecutive days, exactly one is a Sunday. Hence we must count the sets of seven consecutive days which occur in a leap year. Since $366 = 7 \cdot 52 + 2$, there are 52 such sets. But it is possible that one of the two remaining days is also a Sunday. This happens, for example, if the leap year begins on a Saturday, as it did in 1972.

S88S22

Draw $YZ \parallel PQ \parallel AC$ (see diagram). Then, along transversal DC , these three parallels cut off segments YP and PC , and $YP:PC = 1:2$. Hence along transversal XY , $YL:LO = 1:2$. This means that $2/3$ of line segment YO is inside square $MNPQ$. By symmetry, $2/3$ of OX is also inside the square. Adding, we find that $2/3$ of the entire length of XY is inside the square.

S88S23

We wish to maximize the third side of the triangle, so we assume it is the largest side. If its length is c , then we must have $c < \sin x + \cos x$, and we must maximize the function on the right of the inequality. If $S = \sin x + \cos x$, then S is maximal when

$S^2 = \sin^2 x + 2 \sin x \cos x + \cos^2 x = 1 + \sin 2x$ is maximal, and this is clearly greatest when $\sin 2x = 1$ (and $x = \pi / 2$). This makes $S^2 = 2$, so $S = \sqrt{2}$.

Such a triangle is not possible, since the sum of the two of its sides is equal to the third side. But no value of c can be larger than this number.

S88S24

It is not hard to guess the answer, if we note that $\sqrt{1988} = \sqrt{4 \cdot 497} = 2\sqrt{497}$.

To show that this is the only solution, we write: $\sqrt{x} = \sqrt{1988} - \sqrt{y}$, or

$$x = 1988 + y - 2\sqrt{1988y}.$$

Since x is an integer, $1988y$ must be a perfect square, or $2^2 \cdot 7 \cdot 71 \cdot y$ is a perfect square.

This means that y is a multiple of $497 = 7 \cdot 71$, and (since all the arguments are symmetric in x and y), x must be also. Furthermore, both x and y must yield perfect squares when divided by 497 (since $\sqrt{1988y}$ must be an integer, and because of the symmetry in x and y). Writing $x = 497a^2$, $y = 497b^2$, we find from the original equation that $|a| + |b| = 2$, so $a = b = 1$.

Spring 1988 – Senior A – 5 – Solutions

S88S25

The given equation is equivalent to $\log_7 10/x = 5/4$, or $5/4 = \log_7 10/x$, so that $x = 8$.

S88S26

We need to evaluate the given number modulo 100. To do so, we need only keep track of the final two digits. By direct computation, $9^2 = 81$, $9^4 = 61$, $9^8 = 21$, and $9^{10} = 1$, all modulo 10. Hence the exponent of nine can be reduced to its remainder upon division by ten, and the problem can be solved if we find the remainder upon division by 10 of 9^9 . This is not hard: since 9^2 leaves a remainder of 1, so does 9^8 , and 9^9 must leave a remainder of 9. Thus the given number is congruent to 9^9 modulo 100, and using the computational technique given in the second sentence of this solution, the answer is 89.

S88S27

If the number N is represented in decimal notation by $10a + b$, then we have $(10a + b) + (10b + a) = 11(a + b) = K^2$, for some integer K . Since K^2 is a multiple of 11, and 11 is prime, K^2 must also be a multiple of 11^2 . It follows that $a + b = 11$, and there are eight pairs of digits which satisfy this condition. The numbers are 29, 38, 47, 56, 65, 74, 83, and 92.

S88S28

We can compute the probabilities of the various sums in question by finding how many ways the sum can be achieved. In doing this, we may start with the “first two” dice, and see what happens when we throw the third die. In the diagram below, if the first two dice show the sum on the left column, and the third shows the number on the top row, the total will be the number in the table. But we must take into account also the number of ways that two dice can form the sum on the left. These are given as the numerators of the probabilities of the various sums.

For example, from this table, we see that the sum of 7, which can be achieved in 6 ways with two dice, can be formed from three dice in five ways as $6 + 1$, in four ways as $5 + 2$, in three ways as $4 + 3$, in two ways as $3 + 4$, and in one way as $2 + 5$. Thus the sum of 7 can be achieved in $5 + 4 + 3 + 2 + 1 = 15$ ways.

By quick trial and error, one can see that a sum of 10 or of 11 can be formed in 27 different ways. These sums are the most likely outcomes, with equal probabilities of $1/8$.

probability for two dice	sum of two dice	outcome for third die					
		1	2	3	4	5	6
1/36	2	3	4	5	6	7	8
2/36	3	4	5	6	7	8	9
3/36	4	5	6	7	8	9	10
4/36	5	6	7	8	9	10	11
5/36	6	7	8	9	10	11	12

6/36	7	8	9	10	11	12	13
5/36	8	9	10	11	12	13	14
4/36	9	10	11	12	13	14	15
3/36	10	11	12	13	14	15	16
2/36	11	12	13	14	15	16	17
1/36	12	13	14	15	16	17	18

S88S29

No two of the statements can be true, so either they are all false, or exactly one is true. If all are false, then the last statement is true. Hence one statement must be true, making nine of them false (the ninth statement on the list is true).

S88S30

The sum required is the same whether we take the given numbers and do what is described, or if we take their reciprocals first and then form the required sum of products. These sums of products are merely the “elementary symmetric functions” through which the coefficients of a polynomial equation can be expressed in terms of the roots. In fact, if we form the polynomial $P(x) = (x - 1/2)(x - 1/3)(x - 1/4)(x - 1/5)(x - 1/10)$, the answer to this problem is the sum of the absolute values of the coefficients of this polynomial, minus the leading coefficient (which is 1). Since the polynomial is of odd degree, and the signs of its coefficients alternate, this number can be expressed as $-P'(0)$.

Computing directly, we find that

$$P'(x) = (x - 1/2)(x - 1/3)(x - 1/4)(x - 1/10) + (x - 1/2)(x - 1/3)(x - 1/4)(x - 1/10) + \dots + (x - 1/2)(x - 1/3)(x - 1/4)(x - 1/10)$$

$$= 3/2 + 4/3 + 5/4 + 6/5 + 7/6 + 8/7 + 9/8 + 10/9 + 11/10$$

This “telescoping” product is equal to $-11/2$, so the required value is $11/2 - 1 = 9/2$.

The same result can be obtained directly, without discussing a polynomial, by “artificially” forming the product $(1 + 1/2)(1 + 1/3)(1 + 1/4)(1 + 1/10)$. Applying the distributive law shows that this product generates the various sums we need, plus an extra 1. Subtracting this 1 gives the same result as above.