NYCIML
JUNIOR DIVISION
FALL 1991

CONTEST 1

PART 1: 10 MINUTES

F91J1 Compute the sum of the three smallest positive integers, each of which has an odd number of different divisors.

F91J2 The roots of $5x^2 + 6x + 7 = 0$ are A and B. Compute the numerical value of $\frac{1}{A} + \frac{1}{B}$.

PART 2: 10 MINUTES

F91J3 In $\triangle ABC$, $m\angle B = 90^\circ$, AB = 1, $BC = \sqrt{3}$, D is the midpoint of AC, and P is on $\overline{BD}$ with $\overline{AP} \perp \overline{BD}$. Find AP.

F91J4 Compute the area bounded entirely by the graphs of $x + y = 1$ and $\left||x| + |y|\right| = \left||x| - |y|\right|$.

PART 3: 10 MINUTES

F91J5 If $\frac{a}{b} = 5$ and $\frac{b}{c} = 6$, compute the numerical value of the following as a fraction in lowest terms: $\frac{a + 2b + 3c}{2a + 3b + 4c}$.

F91J6 Three distinct integers are chosen randomly from the set {1, 2, 3, ..10}. Compute the probability that their sum is a multiple of three.

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CONTEST 2

PART 1: 10 MINUTES

F91J7 Compute the numerical value of $\left[\dfrac{8000}{\sqrt{2}}\right]$. Note: [x] means the largest integer $\leq$ x.

F91J8 If $A=\sqrt{3}$, $B=\sqrt[3]{5}$, and $C=\sqrt[4]{7}$, arrange A, B, and C in increasing order.

PART 2: 10 MINUTES

F91J9 Compute the sum of the positive integral divisors of 448.

F91J10 The roots of $3x^2+4x+5=0$ are A and B. Compute the numerical value of $\dfrac{A}{B}+\dfrac{B}{A}$.

PART 3: 10 MINUTES

F91J11 A digital clock is one that gives the time as 6:15 or 12:34, etc. If times represent 3 or 4 digit integers, how many multiples of 3 occur in the twelve hour period between 12:00 noon through 11:59 PM inclusively.

F91J12 In convex pentagon ABCDE, AB = 1, BC = 2, CD = 3, DE = 4, $m\angle B=m\angle C=m\angle D=135^\circ$. If x = AE, compute the ordered pair (p, q) where p and q are integers and $x^2=p+q\sqrt{2}$.

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FALL 1991

CONTEST 3

PART 1: 10 MINUTES

F91J13 Compute the closed area bounded entirely by the graph of $\left||x|-|y|\right| = \left||x|+|y|-1\right|$.

F91J14 The digits 1, 2, 3, 4, 5, and 6 are arranged in some order to form a six digit number with these properties:
The left most digit is divisible by 1.
The number formed by the two most left digits is divisible by 2.
The number formed by the three most left digits is divisible by 3.
The number formed by the four most left digits is divisible by 4.
The number formed by the five left most digits is divisible by 5.
The six digit number is divisible by 6.
There are two such numbers, find them.

PART 2: 10 MINUTES

F91J15 Compute how many integers from 100 to 400 inclusive are perfect powers. (perfect squares, perfect cubes, etc.)

F91J16 Four distinct points are chosen randomly on a circle, and randomly labeled A, B, C, and D one letter to a point. Compute the probability that $m\angle ABC + m\angle ADC = 180^\circ$.

PART 3: 10 MINUTES

F91J17 Let x be a positive 2-digit integer and y be the 2-digit integer obtained by reversing the digits of x. Compute the number of x's that make x - y perfect squares.

F91J18 The roots of $2x^2+3x+4=0$ are A and B. Compute the numerical value of $\frac{A^2}{B}+\frac{B^2}{A}$.

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CONTEST 1 - SOLUTIONS

F91J1 In order for an integer to have an odd number of divisors, it must be a perfect square. Thus, the sum of the three smallest of these is 1 + 4 + 9 = 14.

F91J2 Using the theorem regarding sum and product of roots of a quadratic, we have: A + B = -6/5 and AB = 7/5. $\frac{1}{A}+\frac{1}{B}=\frac{A+B}{AB}=\frac{-6/5}{7/5}=-6/7$.

F91J3 Since BD = 1, $\triangle ABD$ is equilateral, and AP is also an altitude making $\triangle BAP$ a 30-60-90 triangle so that $AP=\frac{\sqrt{3}}{2}$.

F91J4 For the absolute value equation: If x = 0, then y can be anything giving us the x-axis. Likewise, if y = 0, then x can equal anything and we get the y-axis. If $x \neq 0 \neq y$, there is no graph since $|x|+|y|=\pm(|x|-|y|)$, from this x or y = 0. Thus, the region is bounded by the two axis and x + y = 1 and the area is .5(1)(1) = 1/2.

F91J5 a = 5b and b = 6c $\rightarrow$ a = 30c.
$$\frac{a+2b+3c}{2a+3b+4c}=\frac{30c+12c+3c}{60c+18c+4c}=\frac{45}{82}$$
Can also be done by letting c = any arbitrary value, say c = 1, then b = 6 and a = 30, etc.

F91J6 Method 1: Considering the numbers mod 3, {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} $\equiv$ $\equiv$ {1, 2, 0, 1, 2, 0, 1, 2, 0, 1} or {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} $\equiv$ {1, -1, 0, 1, -1, 0, 1, -1, 0, 1}. Note there are four 1's and three 0's. The sum of three numbers chosen is congruent to 0 (mod 3) if they are 0, 0, 0 or 1, 1, 1 or -1, -1, -1 or 0, 1, -1. The corresponding number of ways of choosing these are ${}_3C_3, {}_4C_3, {}_3C_3$ and (3)(4)(3) or 1, 4, 1, 36, a total of 42. There are ${}_{10}C_3$ or 120 ways of choosing a number from the original set giving a probability of 42/120 or 7/20.

Method 2: By enumeration (ex. 9 = 1 + 2 + 6 = 1 + 3 + 5 = 2 + 3 + 4 etc.)

Possible sums:	6	9	12	15	18	21	24	27
# of ways:	1	3	7	10	10	7	3	1

gives $\frac{42}{{}_{10}C_3}$.

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CONTEST 2 - SOLUTIONS

F91J7 $\left[\dfrac{8000\sqrt{2}}{2}\right]=\left[4000\sqrt{2}\right]=\left[4000(1.41421)\right]=5656$

F91J8 In order to compare these numbers, raise them to the 12th power:
$A^{12}=3^6=729$
$B^{12}=5^4=625$
$C^{12}=7^3=343$
Thus, in increasing order, they are C, B, A.

F91J9 The prime factorization of 448 is $2^6\cdot 7^1$. Thus, any division of 448 must have the form 2^a OR $7\cdot 2^a$, where a is any integer 0, 1, 2, 3, …,6. Thus, the sum is $(1+2+2^2+....+2^6)+7(1+2+2^2+....+2^6)=8(1+2+2^2+....+2^6)=8(2^7-1)=1016$.

F91J10 Using the theorem regarding sum and product of roots of a quadratic equation, we have A + B = -4/3 and AB = 5/3.
$\dfrac{A}{B}+\dfrac{B}{A}=\dfrac{A^2+B^2}{AB}=\dfrac{(A+B)^2-2AB}{AB}=\dfrac{(A+B)^2}{AB}-2=\dfrac{16/9}{5/3}-2=\dfrac{16}{15}-2=\dfrac{-14}{15}$.

F91J11 Considering the time as 1200-1259, 100-159, 200-259, …, 1100-1159, there are twelve groups of sixty numbers, each containing twenty multiples of 3. Thus, there are 12x20 multiples of 3. 240.

F91J12 $BF=\sqrt{2}$, $DG=2\sqrt{2}$. Thus, $DG>AF$ and H falls to the left of A as shown $EH=EG=GH=EG+DC+CF=2\sqrt{2}+3+\sqrt{2}=3(\sqrt{2}+1)$.
$AH=DG-AF=2\sqrt{2}-(1+\sqrt{2})=\sqrt{2}-1$. The Pythagorean Theorem gives $x^2=9(\sqrt{2}+1)^2+(\sqrt{2}-1)^2=30+16\sqrt{2}$. Thus, the answer is $(30,16)$.

NYCIML
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CONTEST 3 - SOLUTIONS

F91J13 Either $|x|-|y|=|x|+|y|-1 \to y=\pm 1/2$ or $|x|-|y|=-|x|-|y|+1 \to x=\pm 1/2$. Graphically, this is a square with area 1.

F91J14 The digits must alternate odd, then even. The fifth digit must be five. Four cannot be the fourth digit because 14 and 34 are not divisible by 4. Four or six cannot be by the second digit because 1 + 3 + 4 and 1 + 3 + 6 are not divisible by 3. Therefore, the second digit is 2 and the sixth digit is 4. The solutions are 123654 and 321654.

F91J15 Perfect squares: $10^2, 11^2, 12^2, ...20^2$ (11 of them)
Perfect cubes: $5^3, 6^3, 7^3$ (3 of them)
Perfect 4^{th} powers: 4^4 which is $=16^2$ (0 of them)
Perfect 5^{th} powers: 3^5 (1 of them)
Perfect 6^{th} powers: (0 of them)
Perfect 7^{th} powers: 2^7 (1 of them)
Perfect 8^{th} powers: $2^8=16^2$ (0 of them)
Perfect 9^{th}, 10^{th}, etc. powers: (0 of them).
Total: (16 of them). Answer: 16.

F91J16 Once the points have been chosen and labeled, there are three equally likely cases:
Case I: A opposite B.
$m\angle B + m\angle D = 180^\circ$ if $\overline{AC}$ is a diameter. The probability of this is zero.
Case II: A opposite D.
Same as Case I.
Case III: A opposite C.
$\angle C$ is ALWAYS supplementary to $\angle D$ since we have an inscribed quadrilateral.
Thus, in <u>one</u> of the cases, $m\angle B + m\angle D = 180^\circ$. Answer: 1/3.

F91J17 Let $x=10a+b$ so that $y=10b+a$ and $x-y=9(a-b)$. For $9(a-b)$ to be a perfect square, a - b = 0, 1, 4, or 9. Note that $a \neq 0 \neq b$ so $a-b=0 \to a=b \to x=11, 22, 33, ...99.$. If $a=b+1$, then $x=21, 32, ..., 98.$ If $a=b+4$, then $x=51, 62, ..., 95$ and if $a=b+9$ leads to no solutions. Thus, the total number of solutions is 22.

F91J18 Using the theorem regarding sum and product of roots of a quadratic equation, we have $A+B=-3/2$ and AB = 2.

$$\frac{A^2}{B}+\frac{B^2}{A}=\frac{A^3+B^3}{AB}=\frac{(A+B)(A^2-AB+B^2)}{AB}=\frac{(A+B)((A+B)^2-3AB)}{AB}=$$

$$=\frac{-3/2((9/4)-6)}{2}=\frac{45}{16}.$$