NYCIML
SENIOR B DIVISION
FALL 1991

CONTEST 1

PART 1: 10 MINUTES

F91B1 If $i = \sqrt{-1}$, compute the value of $i^1 + i^2 + i^3 + ... + i^{100}$.

F91B2 Ten cards, numbered 1, 2, 3, … 10 are placed face down on a table. One card is drawn at random, its number recorded, and then replaced face down. A card is drawn again. Compute the probability that the number on the second draw exceeds the number on the first draw.

PART 2: 10 MINUTES

F91B3 If the x-intercepts of the curve with the equation $y = x^2 + bx + c$ are 5 and -2, compute the value of c.

F91B4 Compute the real value of (x + y) if $x^3 + 3x^2y + 3xy^2 + y^3 = -27$.

PART 3: 10 MINUTES

F91B5 If $\frac{1}{3}, \frac{1}{x}, \frac{1}{5}$ form an arithmetic progression, compute the value of x.

F91B6 Find the ordered pair (a, b) if the number a, b74, 122 is divisible by 99.

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CONTEST 2

PART 1: 10 MINUTES

F91B7 If $\log_{10} x = 1.5421$, express $10^{3.5421}$ in terms of x.

F91B8 If two roots of $x^3 + px + q = 0$ are 3 and -1, find the third root.

PART 2: 10 MINUTES

F91B9 If [x] means the greatest integer less than or equal to x, find the toot of the equation (x)([x]) = 40.

F91B10 If the probability that it will not rain is the square of the probability that it will rain, find the probability that it will rain.

PART 3: 10 MINUTES

F91B11 If (2, 3), (-1, 9) and (6, k) are collinear, compute the value of k.

F91B12 If $(2+i)^5$ is expressed as a + bi, find the ordered pair (a, b).

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CONTEST 3

PART 1: 10 MINUTES

F91B13 A man can paint a room in three hours, and his son can paint the room in four hours. How long would it take them if they were working together?

F91B14 If $\sqrt[4]{2}$, $\sqrt[6]{2}$, and $\sqrt[12]{2}$ are the first three terms of a geometric progression, compute the value of the fourth term.

PART 2: 10 MINUTES

F91B15 Find the number of terms in $[(x+3y)^2 \cdot (x-3y)^2]^2$ after it is simplified.

F91B16 Find the number of diagonals that can be drawn in a convex polygon with forty sides.

PART 3: 10 MINUTES

F91B17 Successive discounts of 10%, 20%, and 30% are applied to the price of an item. The result is the same as if one discount of x% were applied to the original price of the item. Compute the value of x.

F91B18 In $\triangle ABC$, AB = BC = 1 and $m\angle B = 36^\circ$. Compute the length of $\overline{AC}$.

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CONTEST 4

PART 1: 10 MINUTES

F91B19 What is the degree measure of the angle formed by the hands of a clock when the time is 2:20?

F91B20 The sides of a triangle have lengths 10, 24, 26. Compute the length of the radius of the circle inscribed in this triangle.

PART 2: 10 MINUTES

F91B21 How many ounces of pure water should be added to 20 ounces of a 45% acid solution to make it 30% solution?

F91B22 Find the infinite sum $\frac{1}{7}+\frac{2}{7^2}+\frac{1}{7^3}+\frac{2}{7^4}+\frac{1}{7^5}+\ldots\ldots$.

PART 3: 10 MINUTES

F91B23 Four numbers are chosen from the set {22, 29, 33, 37, 44, 52, 59, 63, 75, 85}. If the product of these numbers is 1, 392, 754, find the four numbers.

F91B24 Write an equation with leading coefficient 1 whose roots are the roots of $x^3+2x-2=0$ each multiplied by 3.

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CONTEST 5

PART 1: 10 MINUTES

F91B25 Find the real root of the equation $x^{3/2} = \frac{8}{27}$.

F91B26 Find all roots of the equation $\sqrt{3-x} = x\sqrt{3-x}$.

PART 2: 10 MINUTES

F91B27 John guesses on every question of a five question true-false quiz. Find the probability of his getting at least 60% correct.

F91B28 $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ are consecutive sides of a regular polygon with n sides, where n > 4. $\overline{AB}$ and $\overline{CD}$ are extended through B and C respectively and meet at Q. In terms of n, find the measure of $\angle Q$.

PART 3: 10 MINUTES

F91B29 Find the smallest whole number which will yield a remainder of 1 when divided by all integers 2 through 10 inclusive.

F91B30 If [x] means the greatest integer less than or equal to x, find the smallest x which satisfies the equation [x] + [2x] + [3x] + [4x] = 15.

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CONTEST 1 - SOLUTIONS

F91B1 The powers of 1 occur in cycles of 4: i, -1, -i,, 1. The sum of these four numbers is 0. Since $i^1 + i^2 + ... + i^{100}$ has exactly 25 cycles, the sum is 0.

F91B2 There are 100 possible outcomes. Of these, there are nine successes if 1 is drawn on the first, eight if 2 is drawn , etc., down to 0 if 10 is drawn on the first. $\frac{9+8+7+....+0}{100} = \frac{45}{100} = \frac{9}{20}$.

F91B3 Since y = 0, 5 and -2 are roots of $x^2 + bx + c = 0$. Since c is the product of the roots, c = -10.

F91B4 $x^3 + 3x^2y + 3xy^2 + y^3 = (x+y)^3 = -27$ so $x + y = -3$.

F91B5 $\frac{1}{x}$ is the average of $\frac{1}{3}$ and $\frac{1}{5} \cdot \frac{1}{x} = \frac{1/3 + 1/5}{2}$. $\frac{1}{x} = \frac{4}{15}$ so $x = 15/4$.

F91B6 Since the number is divisible by 9, the sum of the digits must be divisible by 9. a + b = 2 or a + b = 11. Since the number is divisible by 11, the sum of the even placed digits is equal to the sum of the odd placed digits, or they differ by a multiple of 11.
a + 10 = b + 6, a - b = -4 or a - b =7.
Since a + b = 2 and a - b = -4 produces a =-1, the other pair, a = 9, b = 2 must be the solution.

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CONTEST 2 - SOLUTIONS

F91B7 If $\log_{10} x = 1.5421$, then $10^{1.5421} = x$ and $10^{3.5421} = (10^2)(10^{1.5421}) = (10^2)x = 100x$.

F91B8 Since the coefficient of x^2 is 0, the sum of the roots is 0 and the third root is -2.

F91B9 Since x = 6 produces 36, x = 7 produces 49, the number is between 6 and 7. [x] = 6. 6x = 40, $x = 6\frac{2}{3}$.

F91B10 Since the sum of the probabilities is 1, $p^2 + p = 1$; $p^2 + p - 1 = 0$ and $p = \frac{-1+\sqrt{5}}{2}$. $\frac{-1-\sqrt{5}}{2}$ makes no sense since it is negative.

F91B11 Naming the points A, B, and C, the slope of $\overline{AB}$ must equal the slope of $\overline{BC}$. This means that $\frac{k-9}{6-(-1)} = \frac{9-3}{-1-2}$ and

$$\frac{k-9}{7} = -2 \rightarrow k - 9 = -14 \rightarrow k = -5.$$

F91B12 Using the binomial expression,

$$(2+i)^5 = 2^5 + 5(2^4)i + 10(2^3)(i^2) + 10(2^2)(i^3) + 5(2)(i^4) + i^5$$
$$= 32 + 80i + 80(-1) + 40(-i) + 10 + i = -38 + 41i.$$

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CONTEST 3 - SOLUTIONS

F91B13 Let x represent the time they work together.
$\frac{x}{3}+\frac{x}{4}=1 \rightarrow 4x+3x=12 \rightarrow x=\frac{12}{7}$.

F91B14 Expressing the numbers exponentially, $2^{1/4}, 2^{1/6}, 2^{1/12}$, or $2^{3/12}, 2^{2/12}, 2^{1/12}$. The common ratio is $2^{-1/12}$, and the next number is 2^0 which is 1.

F91B15 The expression equals $(x^2-9y^2)^4$, which will have 4 + 1 or 5 terms since $(x+y)^n$ has $n+1$ terms.

F91B16 The number of diagonals of a polygon with n sides is $\frac{n(n-3)}{2}$. In this case, we have $\frac{40(37)}{2}=740$.

F91B17 If the original cost of the article is x, the discounts will bring the price to 0.9x, 0.72x, 0.504x. Thus, the discount is 49.6%.

F91B18 Let AC = x. Bisecting $\angle C$, and using isosceles triangles gives CD = BD = x. Since $\triangle ABC \sim \triangle ADC$, $\frac{x}{x-1}=\frac{1}{x}$.
$x^2=1-x$
$x^2+x-1=0$
$x=\frac{-1+\sqrt{5}}{2}$
(Reject $\frac{-1-\sqrt{5}}{2}$ since it is negative.)

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CONTEST 4 - SOLUTIONS

F91B19 Two consecutive numbers on a clock are separated by an arc of 30°. At 2:20, the hour hand is one-third of the way to the number three, and the minute hand is on the number four. The central angle therefore contains 50°.

F91B20 Since it is a right triangle, its area is (1/2)(10)(24) = 120.
Since 120 = (1/2)(r)(10) + (1/2)(r)(24) + (1/2)(r)(26), 120 = 30r so that r = 4.

F91B21 The number of ounces of acid in the original solution is equal to the number of ounces of acid in the diluted solution. Therefore, (.45)(20) = .3(20 + x) so that x = 10.

F91B22 Rearranging the terms, we get $\frac{1}{7}+\frac{1}{7^3}+\frac{1}{7^5}+....=\frac{1/7}{1-(1/7)^2}=\frac{7}{48}$
$\frac{2}{7^2}+\frac{2}{7^4}+\frac{2}{7^6}+....=\frac{2/7}{1-(1/7)^2}=\frac{2}{48}$. Thus, the result is $\frac{7}{48}+\frac{2}{48}=\frac{3}{16}$.

F91B23 The number is not divisible by 3 or 5, which eliminates 33, 63, 75, and 85. Since the last two digits are 54, it is an even number, but NOT divisible by 4, eliminating 44 and 52. The remaining numbers are 22, 29, 37, and 59.

F91B24 To multiply the roots by c, multiply the second term by c^1, the second term by c^2, etc. Therefore, as a result, $x^3+0x^2+2x-2=0$ becomes $x^3+3(0x^2)+9(2x)-27(2)=0$, which simplifies to the equation $x^3+18x-54=0$.

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CONTEST 5 - SOLUTIONS

F91B25 Raising both sides of the equation to the 2/3 power, we obtain $(x^{3/2})^{2/3} = (8/27)^{2/3}$. $x = \frac{4}{9}$.

F91B26 Squaring both sides, $3 - x = x^2(3 - x)$ yielding $x = 3, +1, -1$; however, $x = -1$ does not satisfy the original equation. The roots are therefore 1 and 3.

F91B27 He must get <u>at least</u> three correct.
P(3 correct) = $\frac{10}{32}$
P(4 correct) = $\frac{5}{32}$
P(5 correct) = $\frac{1}{32}$.
Thus, the answer is the sum $\frac{16}{32} = \frac{1}{2}$.

F91B28 Since $\angle QBC$ and $\angle QCB$ are exterior angles of a regular polygon, they each have measure $\frac{360^\circ}{n}$. $m\angle Q = 180^\circ - \frac{360^\circ}{n} - \frac{360^\circ}{n} = \frac{180^\circ n - 720^\circ}{n}$.

F91B29 The least common multiple of the integers 2 through 10 is $2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$. The required number is therefore 2521.

F91B30 x is obviously between 1 and 2.
If $x = 1\frac{1}{4}$, the left side of the equation has a value of 11.
If $x = 1\frac{1}{3}$, the left side of the equation has a value of 12.
If $x = 1\frac{1}{2}$, the left side of the equation has a value of 14.
If $x = 1\frac{2}{3}$, the left side of the equation has a value of 15.