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SENIOR A DIVISION
SPRING 1992

CONTEST 1

PART 1: 10 MINUTES

S92S1 Suppose that for positive integers: $f(x) = 1$; $f(2x) = 2f(x) - 1$; $f(2x+1) = 2f(x) + 1$. Find the value of f(100).

S92S2 In triangle ABC, AB = 10, and the median from A has length 8.5. If $\overline{AB}$ is fixed, then all points C that satisfy the above statement form a closed curve. Compute, in terms of π , the area of the region enclosed by this curve.

PART 2: 10 MINUTES

S92S3 A coin is flipped until a head occurs. What is the probability that the number of flips needed is a multiple of 3 or 4?

S92S4 How many ordered pairs of positive integers (a,b) are there such that $\frac{ab}{a+b} = 2^4 \cdot 3^2 \cdot 5^2$.

PART 3: 10 MINUTES

S92S5 If $a_1 = 2, a_{n-1} - 2a_n = 2^{3-2n}$, then find $\sum_{k=1}^{\infty} a_k$.

S92S6 Let $a = \frac{1}{2}(\sqrt{4\sqrt[3]{3} - 7} - 1)$ and $f(x) = (a^2 + a + 2)^x$. Compute the value of $f(f(f(f(f(3)))))$.

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CONTEST 2

PART 1: 10 MINUTES

S92S7 If $x = 2^9 + 2^6 \cdot 3^3 + 2^3 \cdot 3^5 + 3^6$, compute the value of $x^{\frac{2}{3}}$.

S92S8 Find all ordered pairs of natural numbers (a,b), where a<b, for which the greatest common divisor of a and b is 12 and whose product is 277,200.

PART 2: 10 MINUTES

S92S9 Solve the following system of equations for x, y, and z: $\frac{x+y}{xy} = 5$, $\frac{y+z}{yz} = 7$, and $\frac{z+x}{zx} = 6$.

S92S10 Suppose a, b, c are integers such that $a^3 = b^2$, $(a+24)^3 = c^2$, and $abc > 0$. Find all possible values of $a+b+c$.

PART 3: 10 MINUTES

S92S11 The sequence $a_1, a_2 ...$ is defined by: a_n is the units digit of the sum of the first n positive integers. Let $a_1 = 1$. Find the sum of the first 1989 terms of this sequence.

S92S12 A right triangle with area 25/7 is inscribed in a parabola $y = x^2$ with the vertex of the right angle at the origin. What is the length of the longer leg of the triangle?

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CONTEST 3

PART 1: 10 MINUTES

S92S13 In triangle ABC, AB=5, BC=3, and AC=4. D is a point on AB such that AD=2. Find the length of CD.

S92S14 One of the roots of $((1+x)^2)^{\frac{1}{7}} + 3((1-x)^2)^{\frac{1}{7}} = 4(1-x^2)^{\frac{1}{7}}$ is of the form k/1094, where k is a positive integer. Find k.

PART 2: 10 MINUTES

S92S15 Four positive integers are given. Select any three integers, find their arithmetic average, and add this result to the fourth integer. For each different set of three integers chosen the numbers obtained are 29, 23, 21, and 17. Find the smallest of the original integers.

S92S16 On each side of a regular hexagon, a square is constructed interior to the hexagon. If each side of the hexagon has length 1, then the area of the region common to all six squares is $a\sqrt{3}+b$. Find the pair of rational numbers (a,b).

PART 3: 10 MINUTES

S92S17 a, b, c, d, e are 5 different positive digits. Let S be the set of 5-digit integers which can be formed by using each of these digits exactly once. If the average of the numbers in S is 75554.8, what is the smallest number in S?

S92S18 If $\left(\sum_{k=1}^{1990} 2^k\right)^2 - \left(\sum_{k=1}^{1989} 2^k\right)^2 = 2^a + 2^b - 2^c$, where a, b, c are integers with a>b, then compute (a, b, c).

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CONTEST 4

PART 1: 10 MINUTES

S92S19 Two numbers are such that their difference, their sum, and their product are to one another as 1:7:24. Find the product of the two numbers.

S92S20 Find a closed form expression (containing less than three terms) for $2!\cdot 11+4!\cdot 29+6!\cdot 55+...+(2m)!(4m^2+6m+1)$.

PART 2: 10 MINUTES

S92S21 A circle is inscribed in triangle ABC with X, the point of tangency on line segment AC. Also, angle BAC = 90°, $\sqrt{BC+AC}=AB$, and 15AX=XC. Find AX.

S92S22 For a positive integer “a”, the number of positive perfect a^{th} powers divisible by a and less than or equal to b is represented by $f(a,b)$. Compute the value of $f(2,10^5)+f(3,10^5)+f(4,10^5)+f(5,10^5)+...$

PART 3: 10 MINUTES

S92S23 In triangle ABC, the bisectors of the exterior angles at B and C meet at D and the bisectors of angles ABC and ACB meet at E. If BE=3 and BD=5, then what is the maximum value that the product (CD)(CE) can have?

S92S24 Consider the sequence: $a_0=0, a_n=\sqrt{11a_{n-1}+102}$. If the limit exists, what number does a_k approach as k gets very large?

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CONTEST 5

PART 1: 10 MINUTES

S92S25 If $f(x) = x^{1990}$, find the remainder in the division
$$\frac{\left[f(1)+f(2)+f(3)+...+f(1990)\right]}{5}.$$

S92S26 A bus, an hour after starting, has an accident which delays it a half hour, after which it proceeds at ¾ of its former rate and arrives 3 ½ hours late. Had the accident happened 90 miles farther along the line, it would have arrived only 3 hours late. Calculate the length of the trip in miles.

PART 2: 10 MINUTES

S92S27 Compute the value of y if $(\log_3 x)(\log_x 2x)(\log_{2x} y) = \log_x x^2$.

S92S28 A watch loses 2 ½ minutes per day. It is set straight at 1 P.M. on March 15. Let n be the positive correction, in minutes, to be added to the time shown by the watch at a given time. Find the value of n when the watch shows 9 A.M. on March 21.

PART 3: 10 MINUTES

S92S29 Given a 5x12 rectangle with the diagonal forming two triangles, in each triangle a circle is inscribed. Find the distance between their centers.

S92S30 In right triangle ABC, points D and E lie on the hypotenuse AB. CD is equal to 1 and CE is equal to tanx with $0 < x < \frac{\pi}{2}$. Also, AD:EB:AB is equivalent to 1:1:8. If the length of AB can be expressed as ksecx, then compute the value of k.

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CONTEST 1 - SOLUTIONS

S92S1 Proceed naturally to obtain $f(100)=2f(50)-1$, $f(50)=2f(25)-1$, $f(25)=2f(12)+1$, $f(12)=2f(6)-1$, $f(6)=2f(3)-1$, and $f(3)=2f(1)+1$. Now $f(3)=3$, $f(6)=5$, $f(12)=9$, $f(25)=19$, $f(50)=37$, and $f(100)=73$. Note: f is sometimes called the Josephus function and can be easily calculated as follows: Express x in binary code, replace the zeros with (-1)'s, and then go back to base 10. That is, $100_{10}=1100100_2$, thus $f(100)=2^6+2^5-2^4-2^3+2^2-2^1-2^0=73$.

S92S2 Let A be (0,0), B be (10,0), and C be (x, y) in the Cartesian plane. If AM is the median, then M is $\left(\dfrac{x+10}{2},\dfrac{y}{2}\right)$ and we must have $\left(\dfrac{x+10}{2}\right)^2+\left(\dfrac{y}{2}\right)^2=\left(\dfrac{17}{2}\right)^2$. This yields $(x+10)^2+y^2=17^2$. Thus the area is 289π .

S92S3 Let the probability of x flips being needed be P(x). The probability sought in the problem will be

$$\left[P(3)+P(6)+P(9)+...\right]+\left[P(4)+P(8)+P(12)+...\right]-\left[P(12)+P(24)+P(36)+...\right].$$

Note that the subtraction is due to the "12's" counted twice in the sum. We therefore have

$$\left[(1/2)^2(1/2)+(1/2)^5(1/2)+(1/2)^8(1/2)+(1/2)^{11}(1/2)+...\right]+$$
$$+\left[(1/2)^3(1/2)+(1/2)^7(1/2)+(1/2)^{11}(1/2)+...\right]-$$
$$\left[(1/2)^{11}(1/2)+(1/2)^{23}(1/2)+(1/2)^{35}(1/2)+...\right]=$$
$$\frac{\frac{1}{2^3}}{1-\frac{1}{2^3}}+\frac{\frac{1}{2^4}}{1-\frac{1}{2^4}}-\frac{\frac{1}{2^{12}}}{1-\frac{1}{2^{12}}}=\frac{1}{7}+\frac{1}{15}-\frac{1}{4095}=\frac{857}{4095}.$$

S92S4 The given expression can written as:

$ab-2^4 3^5 5^2 a-2^4 3^2 5^2 b=0$
$ab-2^4 3^2 5^2 a-2^4 3^2 5^2 b+2^8 3^4 5^4=2^8 3^4 5^4$
$(a-2^4 3^2 5^2)(b-2^4 3^2 5^2)=2^8 3^4 5^4$

Thus, there are (9)(5)(5)=225 ordered pairs that satisfy the given requirement.

S92S5 Computing several terms, one finds that $a_1 = 2$, $a_2 = 3/4$, $a_3 = 5/16$, $a_4 = 9/64$, $a_5 = 17/256$, we see that $a_n = (2^{n-1}+1)/(4^{n-1})$ and this satisfies the criteria. Now,

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{2^{k-1}+1}{4^{k-1}} = \sum_{k=1}^{\infty}\left[(\tfrac{1}{2})^{k-1} + (\tfrac{1}{4})^{k-1}\right] = = 1/(1-\tfrac{1}{2}) + 1/(1-\tfrac{1}{4}) = 2 + \tfrac{4}{3} = \tfrac{10}{3}.$$

S92S6 Let $a' = \frac{1}{2}(-\sqrt{4\sqrt[3]{3}-7}-1)$. Notice that $a + a' = -1$ and that $a \cdot a' = 2 - \sqrt[3]{3}$. Thus, a and a' are roots of $z^2 + z + 2 - \sqrt[3]{3} = 0$. So $a^2 + a + 2 = \sqrt[3]{3}$ and $f(x) = 3^{x/3}$. Therefore, $f(3) = 3,\ f(f(3)) = 3, etc$.

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CONTEST 2 - SOLUTIONS

S92S7 $(2^3+3^2)^3=(2^3)^3+3(2^3)^2(3^2)+3(2^3)(3^2)^2+(3^2)^3=2^9+2^6\cdot 3^3+2^3\cdot 3^5+3^6$. Thus, $x^{2/3}=(8+9)^2=17^2=289$.

S92S8 The prime factorization of 277,200 is $2^4\cdot 3^2\cdot 5^2\cdot 7\cdot 11$. We need only, then, split the factors into two sets to form two numbers with only the factors 2^2 and 3 in common. This leaves only the three factors 5^2, 7 and 11 to form number pairs, since splitting the two 5's changes the gcd. So we have the following possibilities: 5^2 and $7\cdot 11$; 7 and $5^2\cdot 11$; 11 and $5^2\cdot 7$; 1 and $5^2\cdot 7\cdot 11$. Therefore, the required pairs are 300 and 924; 84 and 3300; 132 and 2100; and 12 and 23100.

S92S9 The given equation can be rewritten as $\frac{1}{x}+\frac{1}{y}=5$, $\frac{1}{y}+\frac{1}{z}=7$, and $\frac{1}{x}+\frac{1}{z}=6$. Letting $\frac{1}{x}=a$, $\frac{1}{y}=b$, and $\frac{1}{z}=c$ yields $a+b=5$, $b+c=7$, and $a+c=6$. Solving, we get $a=2$, $b=3$, and $c=4$, which gives $x=1/2$, $y=1/3$, and $z=1/4$.

S92S10 It must be the case that $a=x^2$ and $a+24=y^2$ for some positive integers x and y. Now $y^2-x^2=24$ yields $(y+x)(y-x)=24\cdot 1,\ 12\cdot 2,\ 8\cdot 3$, or $6\cdot 4$. The integral solutions are $(x,y)=(5,7)$ or $(1,5)$ in which case we have $a=1$ or $a=25$. If $a=1$, then $(b,c)=(1,125)$ or $(-1,-125)$. (Note bc must be positive.) If $a=25$, then $(b,c)=(125,343)$ or $(-125,-343)$. Thus the value of $a+b+c$ can be 127, -125, 493, or -443.

S92S11 By inspection of (a_n), one might guess that $a_n=a_{n+20}$. This can be verified by noting that $\frac{1}{2}(x)(x+1)$ has the same units' digit as $\frac{1}{2}(x+20)(x+21)=\frac{1}{2}(x)(x+1)+20x+210$ for $x\in Z$. Since $a_{20}=0$, let $a_0=0$. Now the answer is the sum of the first 1990 terms of $a_0,a_1,\ldots$. The answer then is $99(a_0+a_1+\ldots+a_{19})+(a_0+\ldots a_9)=99(70)+35=6965$.

S92S12 Let one of the vertices be at (a,a^2). Since the slope from this point to the origin is a, the slope of the other leg is –1/a. The third vertex then will be the intersection of $y=(-1/a)x$ and $y=x^2$. This yields $(-1/a, 1/a^2)$. Now the lengths of the legs are $((-1/a)^2+(1/a^2)^2)^{1/2}$ and $((a)^2+(a^2)^2)^{1/2}$, so we must have $((1/a^2+1/a^4)(a^2+a^4))^{1/2}=50/7$. The left side is equivalent to a+1/a so a=7 or 1/7. The longer leg will have as length the distance from (7,49) to (0,0), which is $35\sqrt{2}$.

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CONTEST 3 - SOLUTIONS

S92S13 Let E be the foot of the perpendicular from D to AC. Now, $\frac{ED}{BC} = \frac{AD}{AB}$, which gives ED= (3)(2/5). Similarly, $\frac{EC}{AC} = \frac{BD}{AB}$, which gives EC=(4)(3/5)=12/5. Now, $CD^2 = ED^2 + EC^2 = 36/25 + 144/25 = 180/25$. Therefore, $CD = (6/5)\sqrt{5}$.

S92S14 Write as $(1+x)^{2/7} - 4(1+x)^{1/7}(1-x)^{1/7} + 3(1-x)^{2/7} = 0$. Factor to get $\left[(1+x)^{1/7} - 3(1-x)^{1/7}\right]\left[(1+x)^{1/7} - (1-x)^{1/7}\right] = 0$. Using the first factor,

$$(1+x)^{1/7} = 3(1-x)^{1/7} \to (1+x) = 3^7 \cdot (1-x) \to 1+x = 2187(1-x) \to x = \frac{1093}{1094}$$

Therefore, k=1093.

S92S15 Solving the system (1/3)(a+b+c)+d=29, (1/3)(b+c+d)+a=23, (1/3)(c+d+a)+b=21, and (1/3)(d+a+b)+c=17 we obtain a=12, b=9, c=3, and d=21. Thus, the answer is 3.

S92S16 It's not hard to see that the region is a hexagon with distance between two opposite sides being $2-\sqrt{3}$. Thus, a side of this hexagon is (1/3)($2\sqrt{3}-3$). Now, the area of a hexagon of side s is $\left(\frac{3\sqrt{3}}{2}\right)s^2$, so the sought area is $\frac{7}{2}\sqrt{3}$ -6 and the required answer is (7/2, -6).

S92S17 There are 5!=120 numbers in S, so each digit occurs in each decimal position 120/5=24 times. Therefore, the sum of the numbers in S is 24(11111)(a+b+c+d+e). The sum is also 120(75554.8). Solve for the sum of the digits to arrive at a+b+c+d+e=34. Since (9+8+7+6+5)=35, the digits (distinct) must be a permutation of (9,8,7,6,4), so the answer is 46789.

S92S18 We have,

$$\left(\sum_{k=1}^{1990} 2^k - \sum_{k=1}^{1989} 2^k\right)\left(\sum_{k=1}^{1990} 2^k + \sum_{k=1}^{1989} 2^k\right) = 2^{1990}\left(2^{1990} + 2\sum_{k=1}^{1989} 2^k\right) =$$

$$= 2^{1990}\left(2^{1990} + 2(2^{1990} - 2)\right) = 2^{1990}\left(2^{1990} + 2^{1991} - 2^{2}\right) = 2^{3980} + 2^{3981} - 2^{1992}.$$

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CONTEST 4 - SOLUTIONS

S92S19 Let the numbers be represented by a and b. Then a+b=7(a-b) and ab=24(a-b). Therefore, 8b=6a and ab=24a-24b=24a-18a=6a. Hence, a(b-6)=0. a=0 does not satisfy the conditions of the problem, so that b=6. Since 6a=8b, a=8. Therefore, ab=48.

S92S20 Note that $(2m)!(4m^2+6m+1)=(2m)![(2m+1)(2m+1)-1]$. We can rewrite the given expression as $2!(3\cdot 4-1)+4!(5\cdot 6-1)+6!(7\cdot 8-1)+\ldots$ $+(2m)!\,[(2m+1)(2m+1)-1]$.
This is a telescopic series that yields
$4!-2!+6!-4!+8!-6!+\ldots(2m+2)!-(2m)! = -2!+(2m+2)! = (2m+2)!$
$=(2m+2)!-2$.

S92S21 Let AX=a. Then AX’’=a, XC=15a, X’C=15a. Let BX’=BX’’=b. Use the Pythagorean Theorem to get $(a+b)^2+(16a)^2=(15a+b)^2$

$$257a^2+2ab+b^2=225a^2+30ab+b^2 \to b=\frac{8a}{7}$$

Use $\sqrt{BC+AC}=AB \to \sqrt{31a+(8/7)a}=\frac{15a}{7}$

$$\frac{225a}{7}=\frac{225a^2}{49}\to a=7.$$

S92S22 Simply proceed directly. $f(2,10^5)$ is the number of elements of $(2^2,4^2,6^2,\ldots)$ which are $\le 10^5$. Thus, $f(2,10^5)$=158. $f(3,10^5)$ is the number of elements of $(3^3,6^3,9^3,\ldots)$, which are $\le 10^5$. Thus, $f(3,10^5)=15$. Now, $f(4,10^5)$ is the number of elements in $(2^4,4^4,6^4,\ldots)$, which are $\le 10^5$, namely 8. The next two are easily determined to be $f(5,10^5)=2$ and $f(6,10^5)=1$. For $x\ge 7$, $f(x,10^5)=0$. Therefore, the answer is 158 + 15 + 8 + 2 + 1 = 184.

S92S23 Let the measure of angles ABC and ACB be a and b, respectively. Notice that angle EBC = a/2 and angle ECB = b/2. Now, angle DBC = $\frac{180-a}{2}$

and angle DCB = $\frac{180-b}{2}$. Thus, angle DBE = $\frac{a}{2}+\frac{180-a}{2}=90^0$ = angle DCE similarity. Thus, $BD^2+BE^2=CD^2+CE^2 \rightarrow 34=CD^2+CE^2$. The geometric-arithmetic inequality is $CD^2+CE^2 \geq 2(CD)(CE) \rightarrow 17 \geq (CD)(CE)$. Thus, the required answer is 17.

S92S24 If the limit exists, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\sqrt{11a_{n-1}+102}.$$

If $L=\lim_{n\to\infty} a_n$, then $L=\sqrt{11L+102}$.

Thus, $L^2-11L-102=0 \rightarrow (L-17)(L+6)=0 \rightarrow L=17$, since $L\neq -6$.

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CONTEST 5 - SOLUTIONS

S92S25 Let $u(x)$ equal the unit digit of $f(x)$. Now, $u(1)=1$, $u(2)=4$, $u(3)=9$, $u(4)=6$, $u(5)=5$, $u(6)=6$, $u(7)=9$, $u(8)=4$, $u(9)=1$, $u(10)=0$. Note that $u(x+10)=u(x)$, thus, the units digit of $f(1)+\dots f(1990)$ has the same units digit as $199\cdot\left(\sum_{k=1}^{10}u(k)\right)=199(45)$, which is 5. Thus, the answer is 0.

S92S26 Since $\log_a b=\dfrac{\log_c b}{\log_c a}$, we have, with base x, $\dfrac{\log x}{\log 3}\cdot\dfrac{\log 2x}{\log x}\cdot\dfrac{\log y}{\log 2x}=2$. $\log y=2\log 3=\log 9$. Therefore, y = 9.

S92S27 The number of minutes in a day is 24x60 = 1440, while the number of minutes as recorded by the watch in a day is $2\frac{1}{2}$ less or $1437\frac{1}{2}$. So the correction factor for this watch is $\dfrac{1440}{1437\frac{1}{2}}=\dfrac{576}{575}$. Designate a minute recorded according to the watch as a “watch-minute” and a day recorded by the watch as a “watch-day”. Since the time interval given is 35/6 watch-days, or (35/6)(24)(60) watch-minutes, we have (35/6)(24)(60) + n = (35/6)(24)(60)(576/575). This yields $n=140\cdot 60((576/575)-1)=14\frac{14}{23}$.

S92S29 Let x be the radius of the circle. The side of length 5 can be divided into segments of lengths x and 5-x. Because triangle ABC is congruent to triangle ACD, AD is also of length 5-x. Similar reasoning on triangle CDE shows that the segment DE must have the length 12-x. Since the diagonal has length 13, (5-x) + (12-x) =13, so x = 2. Let F be the center of the diagonal. Because AF has length 13/2 and AD has length 3, DF must have length 7/2. Using the Pythagorean Theorem, triangle CDF gives the length of CF as $\dfrac{\sqrt{65}}{2}$. The distance between the centers is twice this distance, thus $\sqrt{65}$.

S92S30 Extend the perpendicular from D and E to AC and BC (points O, P, Q, R). Note that DQ = OC = (7/8)AC, CQ = (1/8)BC. Let b = AC and a = BC, and use right triangle CQD to obtain $\frac{49b^2}{64}+\frac{a^2}{64}=CD^2=1$. A similar argument gives $\frac{b^2}{64}+\frac{49a^2}{64}=CE^2=\tan^2 x$. Add to obtain $\frac{50}{64}\left(a^2+b^2\right)=\tan^2 x+1=\sec^2 x$. We wish to find $(a^2+b^2)^{1/2}$, which is $\left(\frac{64}{50}\sec^2 x\right)^{1/2}=\frac{4\sqrt{2}}{5}\sec x$. Thus the answer is $\frac{4}{5}\sqrt{2}$.