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SENIOR A DIVISION
FALL 1992

CONTEST 1

PART 1: 10 MINUTES

F92S1 Let A and B be the points (21, 0) and (0, 20). Let the point P be (a, b). If angle APB is a right angle, then what is the minimum value that a can have.

F92S2 Wade and Scott together choose a number between 1900 and 2000 and wrote each of its positive divisors on ping pong balls. They then took an equal number of balls and gave the left over ball to Cindy who noted that the number on her ball was odd. What are all the possible numbers on Cindy's ball?

PART 2: 10 MINUTES

F92S3 Consider the systems x + (1/y) = 1, y + (1/z) = 2, and z + (1/x) = -3. It must be true that $z = A \pm \sqrt{B}$, with A and B integers. Find (A, B).

F92S4 Consider two very poor machines. One has three parts, two of which must not fail. One part fails with probability 3/5 and each of the other two will fail with a probability p. The second machine fails with probability $p^2 - p + 1$. If the probability of the machines working is the same, find p.

PART 3: 10 MINUTES

F92S5 Find all values of A such that the sum of the cubes and the sum of the squares of the roots of $Ax^2 + 4x + 3 = 0$ have the same value.

F92S6 A superhero flying in a railroad tunnel one day sees a 200 mph train approaching the tunnel. If the hero flies 30 mph and he can reach either end of the tunnel just as the train does, then what fraction of the tunnel has the hero flown through?

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CONTEST 2

PART 1: 10 MINUTES

F92S7 A and B are angles such that cosA + cosB = ½, sinA + sinB = ¼ , and sin2A + sin2B = -27/20. Find sin(A + B).

F92S8 Find four pairs of integral (x, y) such that $x^3 = y^3 + 217$.

PART 2: 10 MINUTES

F92S9 Each side of an equilateral triangle has a semi-circle constructed exterior to the triangle with the side as diameter. Also, the area (in square units) of this figure is numerically equal to the perimeter. Find (a, b) if the side length of the triangle is $\dfrac{a\pi + b\sqrt{3}}{3\pi^2 - 4}$ and a and b contain no radicals.

F92S10 How many different arrangements of 4 letters with no repetitions are in alphabetic order from left-to-right and have as their third letter one of the letters N, Y, C, M, or L?

PART 3: 10 MINUTES

F92S11 Suppose T is a root of $x^4 - x^3 + x^2 - x + 1 = 0$. What is the value of $T^{20} + T^{10} + T^5 + 1$.

F92S12 M and N are points interior to triangle ABC such that angle CAM equals angle BAN and angle CBN equals angle ABM. What is the measure of angle ACB if the sum of angle AMB and angle ANB is 200 degrees?

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CONTEST 3

PART 1: 10 MINUTES

F92S13 Compute $2+\frac{4}{9}+\frac{10}{81}+\frac{28}{729}+\frac{82}{6561}+\ldots$.

F92S14 Suppose that a pair of dice are "loaded" in such a way that the probability of rolling a face with k dots is proportional to k. What is the probability of rolling a sum of seven?

PART 2: 10 MINUTES

F92S15 Find an ordered pair of integers (A, B) such that $2(\sqrt{5}-\sqrt{3}+\sqrt{2})=(\sqrt{5}+\sqrt{3}+\sqrt{2})(\sqrt{A}-\sqrt{B})$.

F92S16 If T^2-1 is a multiple of 256, how many integers T are there such that $1\le T\le 1992$.

PART 3: 10 MINUTES

F92S17 Triangle ABC is inscribed in a circle with AB = 6, BC = 11 and CA = 7. P is on the circle with AP = x, BP = y, and CP = z. For all positions of P on minor arc AB, z = mx + ny. Compute the value of m + n.

F92S18 Compute $\log_{1024}\frac{(\sqrt{3}+i)^{1991}}{(\sqrt{3}+i)}$.

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CONTEST 4

PART 1: 10 MINUTES

F92S19 How many five digit numbers have their digits (no repetitions and from left to right) in decreasing order?

F92S20 Find T if $19x^3 + 92x^2 + 23x + T = 0$ has three roots in geometric progression.

PART 2: 10 MINUTES

F92S21 For what positive integers x is f(x) an integers where $f(x) = \dfrac{x^3 - 98}{x - 5}$.

F92S22 What is the minimum value of $x^2 + y^2$ if $20x + 21y = 1247$.

PART 3: 10 MINUTES

F92S23 Let P be set of all polynomials $f(x)$ with integer coefficients such that the sum of the coefficients is zero. What is the smallest integer $T > 1$ such that T divides $f(1992)$ for all $f(x)$ in P?

F92S24 Find the length of the third median of a scalene triangle with area of $3\sqrt{15}$ and medians of lengths 3 and 6.

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CONTEST 5

PART 1: 10 MINUTES

F92S25 Find all ordered pairs (m, n) such that the roots of $x^2 + mx + n = 0$ are $\frac{m}{3}$ and $\frac{n}{5}$.

F92S26 A quadrilateral ABCD is bisected by diagonal AC area-wise. If A, B, C are (1, 1), (17, 48), and (7, 7), respectively, and D is (a, b), compute the value of a-b.

PART 2: 10 MINUTES

F92S27 Suppose F(x, y) is a function such that $F(x,1) = x^2 + 59x$ and $F(x-3, y-4) = F(x, y)$. Find $F(93,93)$.

F92S28 Let A=1 + $(1/a)$ + $(1/a^2) + (1/a^3)$ + $\ldots$.
Let B = 1 + $(1/b)$ + $(1/b^2) + (1/b^3)$ + $\ldots$. There is one ordered pair (a, b) of positive integers with a>b such that A + B= 15/7. What is this (a, b)?

PART 3: 10 MINUTES

F92S29 Find all $0^\circ \leq \theta \leq 360^\circ$ such that $\sin\theta + \cos\theta = 1 + \sin 2\theta$.

F92S30 Let F (m , n) = (7/2)(m + n) + (3/2)$|m-n|$. For how many pairs of positive integers (m , n) is F (m , n) = 1992.

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CONTEST 1 - SOLUTIONS

F92S1 All the points P are on the circle with diameter AB. The center is (21/2, 10) and the diameter is $\left(20^2+21^2\right)^{1/2}=29$. Thus, the answer is $\frac{21}{2}-\frac{29}{2}=-4$.

F92S2 The number of balls (and thus divisors) is obviously odd since Cindy got the left over ball. So the number chosen originally must be a perfect square since only perfect squares have an odd number of divisors. Thus, they chose $44^2=1936=2^4\cdot 11^2$. Hence, the answer is 1, 11, 121.

F92S3 Solve for x and y. $x=\frac{-1}{z+3}$ and $y=2-\frac{1}{z}$. Substitute and $\frac{-1}{z+3}+\frac{1}{2-\frac{1}{z}}=1$. This yields $z^2+4z-4=0$ after some algebra. Now $z=\frac{-4\pm\sqrt{32}}{2}=-2\pm\sqrt{8}$.

F92S4 P(machine works) = (1-3/5)(1-p)(1-p) + (1-3/5)(1-p)p + (1-3/5)p(1-p) + (3/5)(1-p)(1-p) = (2/5)$(1-p)^2$ + (3/5)$(1-p)^2$ + (4/5)p$(1-p)$ =
= (1/5) p^2 - (6/5)p + 1 = 1- (p^2-p+1). Thus, p = 1 or 5/6.

F92S5 Let the roots be a and b. The $a+b=\frac{-4}{A}$ and $ab=\frac{3}{A}$. Also, $a^3+b^3=(a+b)^3-3ab(a+b)$ and $a^2+b^2=(a+b)^2-2ab$ .Now we want $\left(\frac{-4}{A}\right)^3-3\left(\frac{3}{A}\right)\left(\frac{-4}{A}\right)=\left(\frac{-4}{A}\right)^2-2\frac{3}{A}$. Thus, $3A^2+10A-32=0$ or

$(A-2)(3A+16)=0$ or $A=2$ or $A=\frac{-16}{3}$.

F92S6 Let the length of the tunnel be B, the distance from the train to the tunnel be T, and the fraction be F. Then our two equations are (1-F)B/30 = T/200 and FB/30 = (B+T)/200. This gives us FB/30 = B/200 + (1-F)B/30. Thus, (F- (1-F))/30 =1/200 or 200(2F –1) =30. Therefore, F = 23/40.

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CONTEST 2 - SOLUTIONS

F92S7 1/8 = (cosA + cosB)(sinA + sinB)
= cosAsinA + cosBsinB + cosAsinB + cosBsinA

Thus, the answer is (1/8) – (1/2)(-27/20) = 4/5.

F92S8 We have $(x-y)(x^2+xy+y^2)=1\cdot 217$ or $7\cdot 31$. Suppose $x-y=1$. Thus,
$x^2+xy+y^2=217$ and $(y+1)^2+y(y+1)+y^2=217$
$3y^2+3y+1=217$
$y^2+y-72=0$
$(y+9)(y-8)=0$
$(x,y)=(9,8)$ or $(-8,-9)$
Now, suppose $x-y=7$. Thus, $x^2+xy+y^2=31$
$(y+7)^2+y(y+7)+y^2=31$
$3y^2+21y+49=31$
$y^2+7y+6=0$
$(y+6)(y+1)=0$
$(x,y)=(6,-1)$ or $(1,-6)$

F92S9 Area = $\dfrac{s^2\sqrt{3}}{4}+\dfrac{3}{2}\cdot\left(\dfrac{s}{2}\right)^2\pi = \dfrac{s^2}{8}\left(2\sqrt{3}+3\pi\right)$

Perimeter = $\dfrac{3}{2}s\pi$

Thus, $\dfrac{3}{2}\pi = \dfrac{s}{8}\left(2\sqrt{3}+3\pi\right)$

$$s=\frac{12\pi}{3\pi+2\sqrt{3}}=\frac{12\pi\left(3\pi-2\sqrt{3}\right)}{9\pi^2-12}$$
$$=\frac{4\pi\left(3\pi-2\sqrt{3}\right)}{3\pi^2-4}=\frac{12\pi^2-8\pi\sqrt{3}}{3\pi^2-4}$$
Therefore, $(a,b)=(12\pi,-8\pi)$.

F92S10 N is the 14th letter. We need then two letters before it and one letter after it. Therefore, there are a total of ${}_{13}C_2\cdot 12$ possibilities. Y is the 25th letter and we need a total of ${}_{24}C_2 \cdot 1$ possibilities. There are ${}_{2}C_2 \cdot 23$ possibilities for C, ${}_{12}C_2 \cdot 13$ possibilities for M, and ${}_{11}C_2 \cdot 14$ possibilities for L. Adding these quantities yields $13 \cdot 6 \cdot 12 + 12 \cdot 23 + 23 + 6 \cdot 11 \cdot 13 + 11 \cdot 5 \cdot 14 = 2863$.

F92S11 T is a root of $\frac{(x^5+1)}{x+1} = 0$. Since $T \neq -1$, we know that $T^5 + 1 = 0$ which yields $T^5 = -1$. Thus,
$T^{20} + T^{10} + T^5 + 1 = (T^5)^4 + (T^5)^2 + T^5 + 1 = (-1)^4 + (-1)^2 - 1 + 1 = 2$.

F92S12 Angle AMB = 180 - α - (A-θ)
Angle ANB = 180 - θ - (B - α)
Adding, we obtain,
200 = 360 – A – B = 180 + (180 –A –B)
= 180 + Angle ACB
Therefore, Angle ACB = 20°
It is interesting to note that it does not matter if AN intersects BM.

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CONTEST 3 - SOLUTIONS

F92S13 $\sum_0^{\infty}\frac{3^k+1}{9^k}=\sum_0^{\infty}\left[\left(\frac{1}{3}\right)^k+\left(\frac{1}{9}\right)^k\right]=\frac{1}{1-\frac{1}{3}}+\frac{1}{1-\frac{1}{9}}=\frac{3}{2}+\frac{9}{8}=\frac{21}{8}$.

F92S14 Suppose P_k is the probability of rolling a face with k dots. Then $p_1:p_2:\ldots:p_6=1:2:\ldots:6$ and $p_1+p_2+\ldots+p_6=1$. Thus, $P_k=\frac{k}{21}$. Now, $7=1+6=2+5=3+4=4+3=5+2=6+1$ and the answer is $2(p_1\cdot p_6+p_2\cdot p_5+p_3\cdot p_4)=\frac{2}{21^2}(1\cdot 6+2\cdot 5+3\cdot 4)=\frac{8}{63}$.

F92S15 $\frac{\sqrt{5}-\sqrt{3}+\sqrt{2}}{\sqrt{5}+\sqrt{3}+\sqrt{2}}\cdot\frac{\sqrt{5}-\sqrt{3}+\sqrt{2}}{\sqrt{5}-\sqrt{3}+\sqrt{2}}=\frac{5+\sqrt{10}-\sqrt{6}-\sqrt{15}}{\sqrt{10}+2}=$

$=\frac{5+\sqrt{10}-\sqrt{6}-\sqrt{15}}{\sqrt{10}+2}\cdot\frac{\sqrt{10}-2}{\sqrt{10}-2}=\frac{3\sqrt{10}-3\sqrt{6}}{6}$

Now, $2\frac{\sqrt{5}-\sqrt{3}+\sqrt{2}}{\sqrt{5}+\sqrt{3}+\sqrt{2}}=\sqrt{10}-\sqrt{6}=\sqrt{A}-\sqrt{B}$. Thus, (A, B) = (10, 6).

F92S16 First note that T must be odd. Now, $T^2-1=(T-1)(T+1)$. $T+1$ and $T-1$ are divisible by 2, but only one of them is divisible by 4. Thus, for $\frac{T^2-1}{2^8}$ to be an integer, we must have $2^7\left|T-1\right.$ or $2^7\left|T+1\right.$. Now, $2^7\left|T-1\right.$ if $T=1+128k$. $1\le 1+128k\le 1992$ yields $0\le 128k\le 1991$ or $0\le k\le 15$. $2^7\left|T+1\right.$ if T = -1 + 128k and $1\le -1+128k\le 1992$, which yields $2\le 128k\le 1993$ or $1\le k\le 15$. Therefore, the answer is 16 +15 = 31.

F92S17 By Ptolemy, we have 6z = 7y + 11x or z = (7/6)y + (11/6)x. Thus, m + n = = (7 + 11)/6 = 3.

F92S18 $(\sqrt{3}+i)^{1991}=\left[2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\right]^{1991}=2^{1991}(cis30^\circ)^{1991}=$

$=2^{1991}cis1991\cdot 30^\circ=2^{1991}cis330^\circ=2^{1991}(\cos 330^\circ+i\sin 330^\circ)=$

$$= 2^{1991}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 2^{1991}(\sqrt{3} - i)\,.$$

So $\dfrac{\left(\sqrt{3}+i\right)^{1991}}{\sqrt{3}+i} = 2^{1990}$. We seek α such that $1024^{\alpha} = 2^{10\alpha} = 2^{1990}$, which results in $\alpha = 199$.

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CONTEST 4 - SOLUTIONS

F92S19 Simply pick five digits (without replacement) from $\{0,1,2,3,....,8,9\}$ and put them in the desired order. Thus, we have ${}_{10}C_5 = 252$.

F92S20 Let $k = \left(\frac{-T}{19}\right)^{\frac{1}{3}}$ and the roots be $\frac{a}{r}, a,$ and ar. Now, $\frac{a}{r} \cdot a \cdot ar = k^3$ and $a^3 = k^3$ or $a = k$ and the roots are $\frac{k}{r}, k,$ and kr.
$\frac{k}{r} + k + kr = k\left(\left(\frac{1}{r}\right) + 1 + r\right) = \frac{-92}{19}$. $\frac{k}{r} \cdot k + k \cdot kr + kr \cdot \frac{k}{r} = k^2\left(\frac{1}{r} + r + 1\right) = \frac{23}{19}$.
Now divide and $k = \frac{\frac{23}{19}}{\frac{-92}{19}} = \frac{-1}{4}$. Now, $\left(\frac{-T}{19}\right)^{\frac{1}{3}} = \frac{-1}{4}$, or $\frac{-T}{19} = \frac{-1}{64}$, or $T = \frac{19}{64}$.

F92S21 $f(x) = x^2 + 5x + 25 + \frac{27}{x-5}$. Now we see that $x-5$ must divide 27. Thus, $x-5 = \pm 1$, ± 3, ± 9, and ± 27. The answer then is $x = 2, 4, 6, 8, 14$ and 32.

F92S22 The distance from the line to the origin is
$\frac{|20 \cdot 0 + 21 \cdot 0 - 1247|}{\sqrt{20^2 + 21^2}} = \frac{1247}{29} = 43$. Thus, for all $(x_1, y_1) \in \{(x, y) \mid 20x + 21y = 1247\}$, we have $\sqrt{\left(x_1 - 0\right)^2 + (y_1 - 0)^2} \geq 43$ or $x_1^{\ 2} + y_1^{\ 2} \geq 1849$.

F92S23 Let $f(x) = \sum a_k x^k$. Now $f(1) = \sum a_k = 0$ since $f \in P$. Thus, 1 is a root and $(x-1) \mid f(x)$. Let $x = 1992$, which gives $1991 = 11 \cdot 181 \mid f(1992)$ for all $f \in P$. The answer is 11.

F92S24 The area of the median of a triangle is ¾ the area of the original triangle. Let the unknown median have the length k. Then
$(1/2)(3)(6)\sin\theta = (3/4)(3\sqrt{15})$ or $\sin\theta = \frac{\sqrt{15}}{4}$ where θ is the angle

formed by the sides of lengths 3 and 6. Now $k^2 = 3^2 + 6^2 - 2\cdot 3\cdot 6\cdot\left(1-\left(\frac{\sqrt{15}}{4}\right)^2\right)^{\frac{1}{2}} = 36$ or 54. If k = 6, then the original triangle is not scalene. Hence, $k^2 = 54$ or $k = 3\sqrt{6}$.

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CONTEST 5 - SOLUTIONS

F92S25 We have mn/15 = n and (m/3) + (n/5) = -m or 3n = -20m. Now mn/15 = n which yields n = 0 or m = 15. The answer then is (15, -100) or (0, 0).

F92S26 All triangles AXC which have area equal to that of ABC will have X on (1) the line through B parallel to line AC or (2) the reflection of the line in (1) about line AC. Now since line AC is the x = y, the two lines in (1) and (2) are inverses. Now the line in (1) is y = x + 31, hence the line in (2) is x = y + 31. Therefore, x – y = 31.

F92S27 F(93, 93) = F(90, 89) = F(87, 85) = F(84, 81) = … = F(93-3N, 93-4N) for N a positive (or negative) integer. Notice that $93-4\cdot 23=1$. Thus, F(93, 93)= F(93-$3\cdot 23$,1)= F(24, 1). Now, F(24, 1) = $24^2+59\cdot 24$=24(24+59) = $24\cdot 83$=1992.

F92S28 $A=\dfrac{1}{1-\dfrac{1}{a}}=\dfrac{a}{a-1}$ and $B=\dfrac{b}{b-1}$. Then $A+B=\dfrac{2ab-a-b}{(a-1)(b-1)}$. Hence,

$A+B=\dfrac{15}{7}$.

$14ab-7a-7b=15ab-15a-15b+15$

$ab-8a=8b-15$ or $a=\dfrac{8b-15}{b-8}=8+\dfrac{49}{b-8}$. Now, since $a\in Z^+$, we must have $(b-8)\,|\,49$ or $b-8=\pm 1,\pm 7,$ or ± 49 so $b=1,7,9,15$ or 57 with corresponding $a=1,-41,57,15$ or 9. Thus, the answer is $(57,9)$.

F92S29 First note that $\sin\theta+\cos\theta\geq 0$. Now,
$(\sin\theta+\cos\theta)^2=(1+\sin 2\theta)^2=1+\sin 2\theta$. Thus, $1+\sin 2\theta=0$, 1 or $\sin 2\theta=-1$ or 0. Now $0^\circ\leq 2\theta\leq 720^\circ$. $2\theta=270^\circ,630^\circ$ or $2\theta=0^\circ,180^\circ,360^\circ,540^\circ$. Thus, $\theta=0^\circ,90^\circ,135^\circ,180^\circ,270^\circ,$ or 315°.
$\sin 180^\circ+\cos 180^\circ<0$ and $\sin 270^\circ+\cos 270^\circ<0$. Therefore, we conclude that $\theta=0^\circ,90^\circ,135^\circ,$ or 315°.

F92S30 Suppose $m\geq n$. Then F(n, m) = 5m+2n. If 5m + 2n = 1992 then (m, n) = (398-2t, 5t+1) where t is an integer. (This can be seen by noting that $2n\equiv 2(\operatorname{mod} 5)$ or $n\equiv 1(\operatorname{mod} 5)$.) Now we must have $5t+1\geq 1$ and

$398-2t\geq 1$ or $0\leq t\leq 198$. But we must also have $398\geq 5t+1$ or $t\leq 56$. Hence, $0\leq t\leq 56$ produces 57 solutions. Suppose now that $n\geq m$. These are the same solutions as before, but merely with *m* and *n* switched. Since there is no pair with *m*=*n*, the answer is $57\cdot 2=114$