NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division   CONTEST NUMBER 1

PART I   ***FALL 2009***   ***CONTEST 1***   ***TIME: 10 MINUTES***

F09J1   Matthew is choosing the dimensions of tiles he will use. He can have three tiles, one with length 12, one with length 10 and one with length 15. The widths of the tiles will be 9, 8, and 7. He can pair any width with any length, and each width and length must be used once. Compute the maximum total area of the tiles he chooses.

F09J2   Each student in a group of 12 math students is to be assigned into one of three teams. One team will have three students, one will have four, and one will have five. Compute the number of such assignments.

PART II   ***FALL 2009***   ***CONTEST 1***   ***TIME: 10 MINUTES***

F09J3   Find both ordered pairs *(x, y)* of positive integers such that $x^2 - y^2 = 55$.

F09J4   Compute $\sqrt{20+\sqrt{20+\sqrt{20+\cdots}}}$.

PART III   ***FALL 2009***   ***CONTEST 1***   ***TIME: 10 MINUTES***

F09J5   The sum of five consecutive integers, all of them greater than 1000, is divisible by 15. Compute their least possible sum.

F09J6   In trapezoid $ABCD$ with bases $\overline{AB}$ and $\overline{CD}$, E is on $\overline{AD}$ and F is on $\overline{BC}$ so that $\overline{EF} \,//\, \overline{AB}$ and the areas of $ABFE$ and $EFCD$ are equal. If $AB = 6$, $BC = 9$, $CD = 10$, and $AD = 7$, compute EF.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division CONTEST NUMBER 2

PART I ***FALL 2009*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F09J7 In a circle, chords $\overline{AB}$ and $\overline{CD}$ intersect at E, AE is one less than CE, DE is one more than CE, and BE is twice CE. Find AB.

F09J8 The expression $\sqrt{2}\bullet\sqrt[3]{3}$ can be expressed in the form $\sqrt[n]{m}$, where m and n are positive integers. Compute the minimum value of m.

PART II ***FALL 2009*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F09J9 The diagonals of convex quadrilateral $ABCD$ are perpendicular, $AB = 3$, $BC = 4$, and $CD = 5$. Compute DA.

F09J10 Compute $\sqrt[3]{6+\sqrt[3]{6+\sqrt[3]{6+\cdots}}}$.

PART III ***FALL 2009*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F09J11 The product of 2008 consecutive odd integers is negative, and the largest of them is n. Find the minimum value of n.

F09J12 Compute the coefficient of x^9 in the expansion of $\left(x^2+x+1\right)^9$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division CONTEST NUMBER 3

PART I ***FALL 2009*** ***CONTEST 3*** ***TIME: 10 MINUTES***

F09J13 Find the greatest number of integers, no two of which are the same, whose product is 50.

F09J14 Compute the number of four-digit integers whose digits are all odd or all even.

PART II ***FALL 2009*** ***CONTEST 3*** ***TIME: 10 MINUTES***

F09J15 Find the sum of the coefficients when $\left(1+2x+3x^2+4x^3\right)^3$ is expanded and expressed in simplest form.

F09J16 Two circles intersect at P and Q. Point K is on $\overline{PQ}$, and a line through K intersects one circle at A and B and the other circle at C and D, and the points are in the order $\overline{\mathrm{ACKBD}}$. If $AC = 6$, $CK = 1$, and $KB = 2$, find BD.

PART III ***FALL 2009*** ***CONTEST 3*** ***TIME: 10 MINUTES***

F09J17 Triangle ABC is isosceles with $AC = AB$. A semicircle with diameter $\overline{BC}$ is drawn in the exterior of triangle ABC. The area of the semicircle equals the area of triangle ABC. Compute the tangent of angle ABC.

F09J18 Two parallel cross-sections of a sphere have radii 8 and 12, and the distance between the cross-sections is 2. Compute the radius of the sphere.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division   CONTEST NUMBER 1

Fall 2009 Solutions

F09J1 **301**. Check each of the six possible pairings to find that that the maximum area is $7 \cdot 10 + 8 \cdot 12 + 9 \cdot 15 = 301$. This is an instance of the *Rearrangement Inequality*, which states that given two lists of real numbers of equal length, if the corresponding members of each list are multiplied, the sum of the products is maximum when both lists are placed in order.

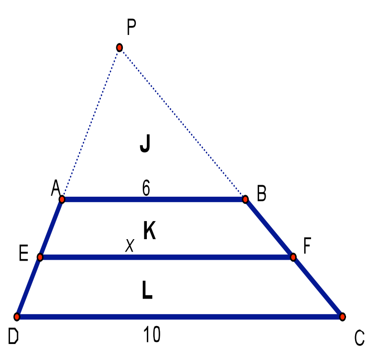
F09J2 **27720**. There are $\binom{12}{3}$ selections for the first 3 students, then $\binom{9}{4}$ selections for the next 4 for a total of $\binom{12}{3}\binom{9}{4} = \frac{12!}{3!9!} \cdot \frac{9!}{4!5!} = \frac{12!}{3!4!5!} = 27720$ possible selections.

F09J3 **(28, 27), (8, 3)**. The given equation is equivalent to $(x+y)(x-y) = 55$. Because $x + y$ and $x - y$ are positive integers, $(x + y, x - y) = (55, 1)$ or $(11, 5)$. Thus $(x, y) = (28, 27)$ or $(8, 3)$.

F09J4 **5**. Let $x = \sqrt{20 + \sqrt{20 + \sqrt{20 + \cdots}}}$. Then $x = \sqrt{20 + x}$, so $x^2 - x - 20 = 0$. Solve to obtain $x = 5$ or $x = -4$. Because $x > 0$, conclude that $x = 5$. Note that this solution depends on the assumption that there is in fact a real number x that equals the given expression.

F09J5 **5025**. Let the numbers be $a - 2$, $a - 1$, a, $a + 1$, $a + 2$. The sum is $5a$, so a must be a multiple of 3. The smallest number is greater than 1000, so $a - 2 > 1000$, or $a > 1002$. The smallest possibility for a is thus 1005, and the sum $5a$ is 5025.

F09J6 $\mathbf{2\sqrt{17}}$. Extend $\overline{DA}$ and $\overline{CB}$ to meet at P. Let $J = [PAB]$, let $K = [ABFE] = [CDEF]$, and let $EF = x$. Then $\frac{x^2}{6^2} = \frac{[PEF]}{[PAB]} = \frac{J+K}{J} = 1 + \frac{K}{J}$, and $\frac{10^2}{6^2} = \frac{[PDC]}{[PAB]} = \frac{J+2K}{J} = 1 + \frac{2K}{J}$, and so $\frac{x^2}{6^2} - 1 = \frac{K}{J} = \frac{1}{2}\left(\frac{10^2}{6^2} - 1\right)$. Solve to obtain $x = 2\sqrt{17}$. Note that EF depends only on the bases.



Challenge: Prove that if $AB = a$ and $CD = b$, then $EF = \sqrt{\frac{a^2+b^2}{2}}$ (which is called the *quadratic mean* of a and b.)

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division CONTEST NUMBER 2

Fall 2009 Solutions

F09J7 **8**. Let $CE = x$. Then $AE = x - 1$, $DE = x + 1$, and $BE = 2x$. Use *Power of a Point* to conclude that $AE \bullet EB = CE \bullet ED$. Thus $(x-1)2x = x(x+1)$. Solve to obtain $x = 0$ or $x = 3$. Because $x > 0$, conclude that $x = 3$, and so $AB = AE + EB = 2 + 6 = 8$.

F09J8 **72**. Let $x = \sqrt{2} \bullet \sqrt[3]{3} = 2^{1/2} \bullet 3^{1/3}$. Then $x^6 = 2^3 \bullet 3^2 = 72$, so $x = \sqrt[6]{72}$. Thus $m = 72$.

F09J9 $\mathbf{3\sqrt{2}}$. Let E be the intersection point of the diagonals AB and CD. Notice that $AB^2 + CD^2 = AE^2 + BE^2 + CE^2 + DE^2 = CE^2 + BE^2 + AE^2 + DE^2$ $= BC^2 + AD^2$. Thus $3^2 + 5^2 = 4^2 + AD^2$, so $AD = 3\sqrt{2}$.

F09J10 **2**. Let $x = \sqrt[3]{6 + \sqrt[3]{6 + \sqrt[3]{6 + \cdots}}}$. Then $x = \sqrt[3]{6 + x}$. Either solve by inspection to obtain $x = 2$, or simplify to obtain $x^3 - x - 6 = 0$, and then $(x-2)(x^2 + 2x + 3) = 0$. Two of the roots are imaginary, and $x = 2$ is the only real root.

F09J11 **1**. Because the product is negative, there must be an odd number of negatives among the 2008 numbers, and so there must be at least one positive. Notice if 1 is chosen as the only positive factor, then the product will be negative. Thus $n = 1$.

F09J12 **3139**. An x^2, an x or a 1 must be chosen from each of the 9 factors of $(x^2 + x + 1)$. Let a be the number of factors of $(x^2 + x + 1)$ from which x^2 is chosen, let b be the number of factors of $(x^2 + x + 1)$ from which x is chosen, and let c be the number of factors of $(x^2 + x + 1)$ from which 1 is chosen. Then $2a + b = 9$, so b must be odd. Thus the possible values for a, b, and c are shown in the table.

a	**4**	**3**	**2**	**1**	**0**
b	**1**	**3**	**5**	**7**	**9**
c	**4**	**3**	**2**	**1**	**0**

The first column indicates that x^2 is chosen four times, x once and 1 four times from the 9 factors of $(x^2 + x + 1)$. There are $\dfrac{9!}{4!1!4!}$ ways of making those choices. Use similar reasoning to conclude that the coefficient of x^9 is

$$\frac{9!}{4!1!4!} + \frac{9!}{3!3!3!} + \frac{9!}{2!5!2!} + \frac{9!}{1!7!1!} + \frac{9!}{0!9!0!} = 630 + 1680 + 756 + 72 + 1 = 3139.$$

Alternate Solution: We can use a Pascal-like Triangle to find coefficients in the expansion of $(x^2 + x + 1)^9$. Since we need only coefficient of x^9, we skip some cells that do not affect the answer. First place 1 at the top. Then each number in the rows below that is equal to the sum of three numbers: the one directly above it and the two that are adjacent to the one directly above.

					1					
				1	**1**	**1**				
			1	**2**	**3**	**2**	**1**			
		1	**3**	**6**	**7**	**6**	**3**	**1**		
	1	**4**	**10**	**16**	**19**	**16**	**10**	**4**	**1**	
1	**5**	**15**	**30**	**45**	**51**	**45**	**30**	**15**	**5**	**1**
…	**…**	**50**	**90**	**126**	**141**	**126**	**90**	**50**	**…**	**…**
…	**…**	**…**	**266**	**357**	**393**	**357**	**266**	**…**	**…**	**…**
…	**…**	**…**	**…**	**1016**	**1107**	**1016**	**…**	**…**	**…**	**…**
…	**…**	**…**	**…**	**…**	**3139**	**…**	**…**	**…**	**…**	**…**

So the answer to the problem is 3139.

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F09J13 **5**. The integer factors of 50 are $\pm 1, \pm 2, \pm 5,$ and ± 25. At most one factor of ± 2 can appear in the product. To obtain the greatest number of integers in the product, it is better to use two ± 5's than one ± 25, and to use as many ± 1's as possible. Note that $50 = -5 \bullet 5 \bullet 2 \bullet (-1) \bullet 1$, so the maximum number of integers in the product is 5.

F09J14 **1125**. If the digits are all odd, there are five choices for each digit for a total of 5^4. If the digits are all even, there are four choices for the first digit and five choices for each of the other three digits for a total of $4 \bullet 5^3$. Thus there are a total of $5^4 + 4 \bullet 5^3 = 625 + 500 = 1125$ of the specified numbers.

F09J15 **1000**. When the *x*'s in the expanded polynomial are replaced by 1's, the result is the sum of the coefficients of the polynomial. Because $\left(1 + 2x + 3x^2 + 4x^3\right)^3$ is equal to the expanded polynomial for all values of *x*, replacing *x* by 1 in this expression will also yield the sum of the coefficients. Thus the requested sum is $(1 + 2 + 3 + 4)^3 = 1000$.

F09J16 **12**. Use *Power of a Point* in each of the two circles to conclude that $AK \bullet KB = PK \bullet KQ = CK \bullet KD$. Thus $7 \bullet 2 = 1 \bullet KD$, so $KD = 14$, and then $BD = 12$.

F09J17 $\boldsymbol{\pi / 2}$ .Let O be the center of the semicircle, let r be the radius and let $h = OA$. The area of the semicircle is $\frac{1}{2}\pi r^2$, and the area of ABC is $\frac{1}{2}(2r)h = rh$. Thus $\frac{1}{2}\pi r^2 = rh$, so $h = \pi r / 2$, and then $\tan ABC = h / r = \pi / 2$.

F09J18 $\boldsymbol{\sqrt{505}}$. Let a and b be the distances from the center to the cross-sections with radii 8 and 12, respectively. For each cross-section, consider the right triangle whose vertices are a point where the cross-section meets the surface of the sphere, the center of the cross-section, and the center of the sphere. Use the Pythagorean Theorem to conclude that $8^2 + a^2 = r^2 = 12^2 + b^2$, so $80 = a^2 - b^2 = (a-b)(a+b)$. If the two cross-sections are on the same side of the sphere's center, then $a - b = 2$, and if they are on opposite sides, then $a + b = 2$. If a – $b = 2$, then $a + b = 40$, and so $(a, b) = (21, 19)$. But if $a + b = 2$, then $a - b = 40$, which is impossible. Thus $(a, b) = (21, 19)$, and $r = \sqrt{505}$.