New York City Interscholastic Math League
Senior A Division   Contest Number 1

Part I   Fall 2009   Contest 1   Time: 10 Minutes

F09A1 Compute the ordered triple (a, b, c) such that

$$a + b + c = 4,$$
$$a + 2b + 4c = 9,$$
$$a + 3b + 9c = 16.$$

F09A2 A regular polygon has the property that it remains unchanged when it is rotated by 25° around its center. What is the smallest number of sides this polygon could have?

Part II   Fall 2009   Contest 1   Time: 10 Minutes

F09A3 Compute the number of *nonnegative* integral solutions (x, y, z) to the equation $2x + 3y + 6z = 30$.

F09A4 The number 9 is the smallest positive integer with the property that if you add or subtract either 2 or 4 from it, you get a prime number (one of 5, 7, 11 and 13). The second-smallest number with this property is 15. Compute the third-smallest number with this property.

Part III   Fall 2009   Contest 1   Time: 10 Minutes

F09A5 Compute the smallest positive integer n such that the sum $1 + 2 + 3 + \ldots + n$ is divisible by 12.

F09A6 Two concentric circles are drawn, one of radius 1 and the other of radius 3. From a point P on the larger circle, two tangent lines to the smaller circle are drawn. These lines intersect the larger circle at points Q and R. Compute the area of $\triangle PQR$.

New York City Interscholastic Math League
Senior A Division   Contest Number 2

Part I   Fall 2009   Contest 2   Time: 10 Minutes

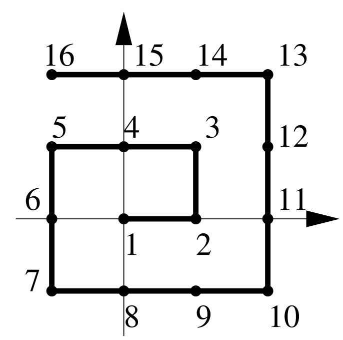
F09A7 A collection of five integers, not necessarily distinct, has median 6 and unique mode 5. Compute the smallest possible value for the mean of the five numbers.

F09A8 Alejandro has a fair die whose sides are labelled $2, 2, 4, 4, 5, 5$. Martina has a fair die whose sides are labelled $1, 1, 3, 3, 6, 6$. They play the following game: both roll their dice simultaneously, and the one with the higher number wins. Compute the probability that Alejandro wins the game.

Part II   Fall 2009   Contest 2   Time: 10 Minutes

F09A9 What point on the line $2y + 3x = 5$ is closest to the origin?

F09A10 In the diagram shown, every point in the plane with integer coordinates is labelled with a positive integer by a "rectangular spiral" process that begins at the origin and works its way out. What is the label on the point $(20, 9)$?



Part III   Fall 2009   Contest 2   Time: 10 Minutes

F09A11 There is a unique pair (m, n) of positive integers such that $n = m + 10$ and n is a divisor of $3m - 1$. Compute this ordered pair.

F09A12 Compute all complex numbers x such that

$$(x^2 - 4x + 6)^2 - 4(x^2 - 4x + 6) + 6 = x.$$

New York City Interscholastic Math League
Senior A Division — Contest Number 3

Part I — Fall 2009 — Contest 3 — Time: 10 Minutes

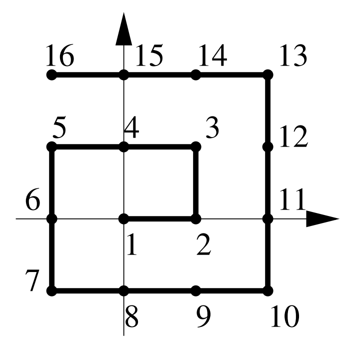
F09A13 Shaun walked at a constant speed of 4 miles per hour to go pick up his bike, after which he bicycled at a constant speed of 12 miles per hour to his destination. If Shaun biked for 10 minutes and had an average speed for the whole trip of 8 miles per hour, what distance in miles did he walk?

F09A14 Compute the smallest positive integer n such that the sum $1+2+3+\ldots+n$ is divisible by 96.

Part II — Fall 2009 — Contest 3 — Time: 10 Minutes

F09A15 Solve the equation $2z=(1+i)\overline{z}+4$ for z in the complex numbers. (Here $i=\sqrt{-1}$ is the imaginary unit and $\overline{z}$ denotes the complex conjugate of z.) Express your answer in the form $a+bi$, where a and b are real numbers.

F09A16 In the diagram shown, every point in the plane with integer coordinates is labelled with a positive integer by a "rectangular spiral" process that begins at the origin and works its way out. What are the coordinates of the point labelled 2009?



Part III — Fall 2009 — Contest 3 — Time: 10 Minutes

F09A17 Compute $\sec^2\left(\arctan\left(\frac{\sqrt[4]{12}}{2}\right)\right)$.

F09A18 The polynomial $f(x)=x^4+ax^3+bx^2+cx+d$ has four distinct complex zeros. Each zero of $f(x)$ is the sum of one of the zeros of x^2+2x+3 and one of the zeros of x^2-4x-3. Compute $f(0)$.

New York City Interscholastic Math League
Senior A Division Contest Number 4

Part I Fall 2009 Contest 4 Time: 10 Minutes

F09A19 Compute the value of $\tan t + \cot t$ when $\sin t + \cos t = \frac{3}{5}$.

F09A20 The sequence a_n is defined by $a_0 = 0$ and $a_{n+1} = 2a_n + 1$ for all $n \geq 0$. Compute the value of the infinite series $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{3^n} = \frac{a_0}{1} + \frac{a_1}{3} + \frac{a_2}{9} + \ldots$.

Part II Fall 2009 Contest 4 Time: 10 Minutes

F09A21 The region R consists of the set of points in the plane above the x-axis and below the graph of the function $y = 3 - |2 - |x||$. Compute the area of R.

F09A22 Compute all integers n such that n is exactly twice the number of positive divisors of n (counting both 1 and n as divisors).

Part III Fall 2009 Contest 4 Time: 10 Minutes

F09A23 Compute the coefficient of x^3 in $(x^2 + 2x + 1)^{10}$.

F09A24 Starting at the point (x, y), a *step* is a move to one of the points $(x+1, y)$ or $(x, 1-y)$. A *walk* is a sequence of steps that does not visit any point more than once. Compute the number of walks that begin at $(0, 0)$ and end at $(10, 0)$.

New York City Interscholastic Math League
Senior A Division   Contest Number 5

Part I   Fall 2009   Contest 5   Time: 10 Minutes

F09A25 An informal poll of top university math departments found that 11% of mathematicians surveyed were absentminded and brilliant, 13% were brilliant and incomprehensible, 19% were incomprehensible and absentminded, and 6% were all three. What percentage of mathematicians surveyed had exactly two of these three properties?

F09A26 What point on the circle $y^2 - 4y + x^2 - 2x - 4 = 0$ is closest to the origin?

Part II   Fall 2009   Contest 5   Time: 10 Minutes

F09A27 Compute all ordered triples (a, b, c) of real numbers such that

$$\begin{aligned} a^2 + 4b^2 &= 1 \\ 4b^2 + 9c^2 &= 1 \\ 6ac &= 1 \end{aligned}$$

and $a > 0$.

F09A28 In $\triangle ABC$, $AB = 6$, $AC = 8$ and $BC = 10$. Equilateral $\triangle APQ$ has points P and Q on segment BC. Compute the area of $\triangle APQ$.

Part III   Fall 2009   Contest 5   Time: 10 Minutes

F09A29 Compute two more than the positive square root of the answer to this question.

F09A30 Compute the smallest positive integer n such that 2009 can be written as a sum of n *odd* perfect squares. (Repetition is allowed; for example, $13 = 9 + 1 + 1 + 1 + 1$ and rearrangements of this sum are the only way to write 13 as a sum of 5 odd perfect squares.)

New York City Interscholastic Math League
Senior A Division Contest Number 1 Solutions

F09A1. $(\mathbf{1}, \mathbf{2}, \mathbf{1})$. Notice that the polynomial $f(x) = cx^2 + bx + a$ satisfies $f(1) = 4$, $f(2) = 9$ and $f(3) = 16$. Then we can see that $f(x) = (x+1)^2 = x^2 + 2x + 1$ and so $(a, b, c) = (1, 2, 1)$.

Alternatively, subtract the first equation from the latter two to get $b + 3c = 5$ and $2b + 8c = 12$. Now subtract twice the first of these equations from the second to get $2c = 2$, so $c = 1$. Now we may back-substitute to get $b = 2$ and $a = 1$.

F09A2. **72**. An n-gon is preserved under rotation by any multiple of $\frac{360^\circ}{n}$, so we need to find the smallest positive integer n such that there exists an integer m satisfying $\frac{360m}{n} = 25$ or $72m = 5n$. Since 72 and 5 are relatively prime, the smallest solution has $n = 72$.

F09A3. **21**. Since $3y$, $6z$ and 30 are all divisible by 3, we must have $x = 3m$ and similarly $y = 2n$ for some integers m and n. Then $6m + 6n + 6z = 30$ or $m + n + z = 5$. Now use the method of "stars and bars" to compute that the answer is $\binom{5+3-1}{5} = \binom{7}{5} = \frac{7!}{5!2!} = 21$.

Alternatively, we may use an organized listing procedure on either the original or simplified equation: when $z = 5$ there is one solution, when $z = 4$ there are two, when $z = 3$ there are three, and so on. Thus the answer is $1 + 2 + 3 + 4 + 5 + 6 = 21$.

F09A4. **105**. Suppose that our answer is n, and note that n must be odd. We are given that $n-4, n-2, n+2$ and $n+4$ are all prime. Note that 5 must divide one of any five consecutive odd integers (modulo five, they are congruent to $n+1, n+3, n, n+2$ and $n+4$, and exactly one of these is divisible by 5) and similarly 3 must divide at least one of these numbers. Since $n \pm 2, n \pm 4$ are prime, none of them may be divisible by 3 or 5, and so n must be divisible by both. Now we may quickly check that 45 and 75 do not satisfy the desired conditions (49 and 77 are not prime) but that 105 does. The next few such numbers are $195, 825, 1485, 1875, \ldots$. (Note that it is not known that there are infinitely many such numbers, since that would imply that there are infinitely many twin primes.)

F09A5. **8**. We can compute by hand that the first number of the form $1 + 2 + \ldots + n$ divisble by 12 is 36, when $n = 8$.

To find this systematically, we note that 12 must divide $\frac{n(n+1)}{2}$ and so 24 must divide $n(n+1)$. Since n and $n+1$ are relatively prime, this means that 24 divides n, that 24 divides $n+1$, that 3 divides n and 8 divides $n+1$, or that 8 divides n and 3 divides $n+1$. The smallest n satisfying these conditions are respectively $n = 24$, $n = 23$, $n = 15$ and $n = 8$, and the smallest of these is the answer, 8.

F09A6. $\mathbf{\dfrac{64\sqrt{2}}{9}}$. Let the center of the circles be O and let M be the midpoint of $\overline{QR}$. Triangle $\triangle OPQ$ is isosceles with equal sides $PO = OQ = 3$ and altitude from O of length 1. Then by the Pythagorean Theorem, $PQ = 2\sqrt{3^2 - 1^2} = 4\sqrt{2}$. Let $OM = a$ and $QM = b$. Then applying the Pythagorean Theorem in $\triangle OQM$ gives $a^2 + b^2 = 3^2$ and in $\triangle PQM$ gives $(3+a)^2 + b^2 = (4\sqrt{2})^2$. Thus $18 + 6a = 32$ so $a = \frac{7}{3}$ and $b = \frac{4\sqrt{2}}{3}$. Then the area in question is $A(\triangle PQR) = \frac{1}{2} \cdot (a+3) \cdot 2b = \frac{16}{3} \cdot \frac{4\sqrt{2}}{3} = \frac{64\sqrt{2}}{9}$.

Alternatively, let S be the point of tangency of PQ with the smaller circle. Note that $\triangle POS \sim \triangle PQM$ and $A(\triangle POS) = \sqrt{2}$, so $A(\triangle PQM) = \sqrt{2} \cdot \frac{PO^2}{PQ^2} = \frac{32\sqrt{2}}{9}$.

New York City Interscholastic Math League
Senior A Division — Contest Number 2 Solutions

F09A7. $\mathbf{\frac{31}{5}}$. Let the five integers be $t_1 \le t_2 \le t_3 \le t_4 \le t_5$. Then $t_3 = 6$. Since the mode is 5, at least two of the members of the sequence must be equal to 5; since $5 < 6$ we must have $t_1 = t_2 = 5$. Because the mode is unique, t_3, t_4 and t_5 must all be different. To make the mean as small as possible, choose $t_4 = 7$ and $t_5 = 8$, which gives a mean of $\frac{5+5+6+7+8}{5} = \frac{31}{5}$.

F09A8. $\mathbf{\frac{5}{9}}$. Alejandro rolls 2 with probability $\frac{1}{3}$, in which case he wins if and only if Martina rolls a 1, with probability $\frac{1}{3}$. Alejandro rolls 4 or 5 with probability $\frac{2}{3}$, in which case he wins if Martina rolls 1 or 3, with probablity $\frac{2}{3}$. Thus, Alejandro wins with probability $\frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{3} = \frac{5}{9}$.

F09A9. $\mathbf{\left(\frac{15}{13}, \frac{10}{13}\right)}$. The given line has slope $-\frac{3}{2}$. The point P on this line closest to the origin O is the point such that OP is perpendicular to the given line. Thus, line OP must have slope $\frac{2}{3}$. If $P = (x_0, y_0)$, the slope of OP is $\frac{y_0}{x_0}$, so $y_0 = \frac{2}{3}x_0$. Also this point is on the given line, so $2y_0 + 3x_0 = 5$. Solving this system of two equations gives $x_0 = \frac{15}{13}$ and $y_0 = \frac{10}{13}$, making our answer $\left(\frac{15}{13}, \frac{10}{13}\right)$.

F09A10. **1550**. Observe that when our path reaches the point $(n, -n)$ for any positive integer n, it has visited every point in a square with $2n + 1$ dots per side centered at the origin, and no others. Thus, when it visits the point $(19, -19)$, it has filled out a square containing 39^2 dots. From there, it steps right to the point $(20, -19)$ and then proceeds directly up to the point $(20, 9)$, passing through $9 - (-19) + 1 = 29$ more points. It follows that the label on the point $(20, 9)$ is exactly $39^2 + 29 = 1550$.

F09A11. $(\mathbf{21}, \mathbf{31})$. We have that $3m - 1 < 3m < 3n$ is divisible by n, so either $n = 3m - 1$ or $2n = 3m - 1$. With $n = m + 10$ the first equation gives $2m = 11$, with no integral solutions, while the second gives $m = 21$ and so $n = 31$.

Alternatively, we have that $\frac{3m-1}{n} = \frac{3m-1}{m+10}$ is an integer. This is true if and only if $3 - \frac{3m-1}{m+10} = \frac{31}{m+10}$ is an integer, so we must have $m + 10$ divides 31. Since $m > 0$ is an integer, the only solution is $m = 21$ and so $n = 31$.

F09A12. $\mathbf{2, 3, \frac{3 + i\sqrt{3}}{2}, \frac{3 - i\sqrt{3}}{2}}$ (all four values are required!). Let $f(x) = x^2 - 4x + 6$. Then we are trying to solve the equation $f(f(x)) = x$. Note that any solution to the equation $f(x) = x$ will also be a solution to our equation, so we begin by solving $x^2 - 4x + 6 = x$. We may rewrite this as $x^2 - 5x + 6 = 0$, with roots 2 and 3. It follows that these must be roots of our original equation. Expanding that equation out gives $x^4 - 8x^3 + 24x^2 - 33x + 18 = 0$, and we divide out by the known factors $x - 2$ and $x - 3$, leaving $x^2 - 3x + 3 = 0$. Finally, we apply the quadratic formula to get the other two roots, $x = \frac{3 \pm i\sqrt{3}}{2}$.

Alternatively, let $y = x^2 - 4x + 6$. Then $y^2 - 4y + 6 = x$ and $x^2 - 4x + 6 = y$. One way to solve this system is to subtract one of these equations from the other and factor. A second method is to rewrite these equations as $(y - 2)^2 = x - 2$ and $(x - 2)^2 = y - 2$ and substitute.

New York City Interscholastic Math League
Senior A Division   Contest Number 3 Solutions

F09A13. $\mathbf{\frac{2}{3}}$. Let the distance in question be d. Shaun biked for $\frac{1}{6}$ hour at 12 miles per hour, so his total distance is $2+d$. He spent $\frac{d}{4}$ hours walking and $\frac{1}{6}$ hours biking, so his total time is $\frac{d}{4}+\frac{1}{6}$. Thus his average speed is

$$8 = \frac{2+d}{\frac{d}{4}+\frac{1}{6}}.$$

Multiplying through, this gives $2d+\frac{4}{3} = d+2$ and so $d = \frac{2}{3}$.

Alternatively, since 8, the speed for the whole trip, is the average of 4 and 12, the speeds on the two sections, Shaun must have walked and biked for equal amounts of time. Thus Shaun walked for $\frac{1}{6}$ hour at 4 miles per hour and so walked $\frac{2}{3}$ miles.

F09A14. **63**. We have that 96 divides $\frac{n(n+1)}{2}$ and so $192 = 2^6 \cdot 3$ divides $n(n+1)$. Since n and $n+1$ are relatively prime, either 192 divides one of them or 2^6 divides one and 3 divides the other. One can quickly check that the smallest number satisfying any of these conditions is 63.

F09A15. $\mathbf{6+2i}$. We have $2(a+bi) = (1+i)(a-bi)+4$ and so $(a-b-4)+(3b-a)i = 0$. A complex number is 0 if and only if its real and imaginary parts are 0, so $a-b-4 = 0$ and $3b-a = 0$. Solving this system gives $b = 2$ and $a = 6$, so the unique solution is $z = 6+2i$.

F09A16. $(\mathbf{6}, \mathbf{-22})$. We have that $44^2 = 1936 < 2009 < 45^2 = 2025$. The spiral fills a square of side-length 45 when it reaches the point $(22, -22)$. At that moment, it has just taken 44 steps directly to the right. So it labelled a point 2009 exactly $2025-2009 = 16$ units to the left of $(22, -22)$, and our answer is $(6, -22)$.

F09A17. $\mathbf{1+\frac{\sqrt{3}}{2}}$ or $\mathbf{\frac{2+\sqrt{3}}{2}}$. Let $t = \frac{\sqrt[4]{12}}{2}$ and let $\theta = \arctan t$. Then $\tan\theta = t$. Since $\sec^2 x = 1+\tan^2 x$ for all x, this implies $\sec^2\theta = 1+t^2 = 1+\frac{\sqrt{3}}{2}$.

F09A18. **72**. Let the zeros of $g(x) = x^2+2x+3$ be r_1 and r_2 and the zeros of $h(x) = x^2-4x-3$ be s_1 and s_2. Then $f(x) = (x-r_1-s_1)(x-r_1-s_2)(x-r_2-s_1)(x-r_2-s_2)$. Note that by grouping the first and second pair of factors we can write this as $f(x) = h(x-r_1)h(x-r_2)$ and so $f(0) = h(-r_1)h(-r_2) = (r_1^2+4r_1-3)(r_2^2+4r_2-3)$. Expand this to get

$$f(0) = (r_1r_2)^2 + 4r_1r_2(r_1+r_2) - 3(r_1^2+r_2^2) + 16r_1r_2 - 12(r_1+r_2) + 9.$$

Since r_1 and r_2 are the zeros of $g(x)$, we have $r_1r_2 = 3$ and $r_1+r_2 = -2$ and so also $r_1^2+r_2^2 = (r_1+r_2)^2 - 2r_1r_2 = -2$. (Note that r_1 and r_2 are complex numbers, so this is okay; if they were real numbers, the sum of two squares couldn't be nonnegative.) Finally, substituting all these values gives $f(0) = 3^2 + 4\cdot 3\cdot(-2) - 3(-2) + 16\cdot 3 - 12(-2) + 9 = 72$.

Alternatively, it is possible to solve the two quadratic equations, add their zeros, and multiply everything out. It helps to recognize that you only need to compute the constant coefficient of f, not the entire polynomial.

Challenge: prove that if the two quadratic polynomials are x^2+mx+n and x^2+px+q then the resulting answer is $(n-q)^2+(m+p)(np+mq)$.

New York City Interscholastic Math League
Senior A Division Contest Number 4 Solutions

F09A19. $-\mathbf{\frac{25}{8}}$. From $\sin t + \cos t = \frac{3}{5}$ we have $\sin^2 t + 2\sin t\cos t + \cos^2 t = \frac{9}{25}$ and so $\sin t\cos t = \frac{1}{2}(\frac{9}{25} - 1) = -\frac{8}{25}$. Then

$$\tan t + \cot t = \frac{\sin^2 t}{\sin t\cos t} + \frac{\cos^2 t}{\sin t\cos t} = \frac{1}{\sin t\cos t} = -\frac{25}{8}.$$

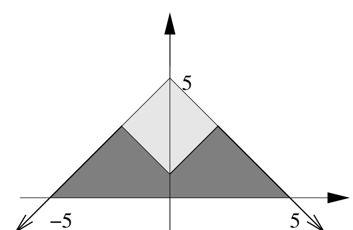
F09A20. $\mathbf{\frac{3}{2}}$. It's not hard to prove by induction that $a_n = 2^n - 1$, after which our series becomes

$$\sum_{n=0}^{\infty} \frac{2^n - 1}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n - \sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-\frac{2}{3}} - \frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

Alternatively, write $S = \displaystyle\sum_{n=0}^{\infty} \frac{a_n}{3^n}$. Then

$$S = \sum_{n=1}^{\infty} \frac{a_n}{3^n} = \sum_{n=1}^{\infty} \frac{2a_{n-1} + 1}{3^n} = \frac{2}{3}S + \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{2}{3}S + \frac{1}{2}.$$

Thus $\frac{S}{3} = \frac{1}{2}$ and $S = \frac{3}{2}$.



F09A21. **17**. The region R consists of a isosceles right triangle of hypotenuse 10 with a square of diagonal 4 removed. Thus, it has area $25 - 8 = 17$.

F09A22. $\mathbf{8, 12}$. Write $n = 2^a \cdot 3^b \cdot 5^c \cdots$. The number of divisors of n is $(a+1)(b+1)(c+1)\cdots$, and we are given that twice this number is exactly equal to n. Note that $2^a \geq a+1$ for all nonnegative integers a, $3^b \geq b+1$ for all nonnegative integers b, and for $p \geq 5$ and $c \geq 1$ we have $p^c > 2(c+1)$. It follows that if any prime $p \geq 5$ divides n that $n = 2^a \cdot 3^b \cdot 5^c \cdots > 2(a+1)(b+1)(c+1)\cdots$, a contradiction, so we must have $n = 2^a 3^b$. Then we need to solve the equation $2^a 3^b = 2(a+1)(b+1)$. If $b \geq 2$ then $3^b > 2(b+1)$, so we must have either $b = 0$ or $b = 1$. If $b = 0$, we are solving $2^a = 2(a+1)$, with unique integral solution $a = 3$, while if $b = 1$ we are solving $3 \cdot 2^a = 4(a+1)$ with unique integral solution $a = 2$. These give n-values of 8 and 12, our answers.

Alternatively, write $n = 2k$ for some positive integer k, where k is also the number of divisors of n. Check that none of the values $k = 1, 2$ or 3 ($n = 2, 4$ or 6) are solutions. Now consider $k \geq 4$. We have that n itself is a divisor of n and that $k = \frac{n}{2}$ is the next-largest divisor of n, so none of the $k-1$ numbers between k and n, exclusive, are divisors of n. Also, $k-1$ cannot be a divisor of n: if $k-1$ is a divisor of $2k$ then it is also a divisor of $2k - 2(k-1) = 2$, so $k \leq 3$, but we have $k \geq 4$. This means that we have already found k values between 1 and n, inclusive, that are *not* divisors of n. Thus all of the other k values between 1 and n must be divisors of n. In particular, the relatively prime integers $k-2$ and $k-3$ are both divisors of n, and so their product $(k-2)(k-3)$ is, as well. Thus

$(k-2)(k-3) \le 2k$ and so $(k-1)(k-6) \le 0$. Together with $k \ge 4$, this leaves only the values $k = 4, 5$ and 6 to check; we see that $k = 4$ and 6 work but that $k = 5$ does not, so the answer is 8 and 12.

F09A23. **1140**. We have $(x^2 + 2x + 1)^{10} = (x + 1)^{20}$ and so the coefficient of x^3 in this expression is $\binom{20}{3} = \frac{20 \cdot 19 \cdot 18}{3!} = 1140$ by the Binomial Theorem.

Alternatively, we can make a term of degree three by choosing one term of degree two (10 choices, each with a coefficient of 1) and one term of degree one (9 choices, each with a coefficient of 2) and the rest of degree zero (1 choice with a coefficient of 1), for a total contribution of $10 \cdot 9 \cdot 2 = 180$, or by choosing three terms of degree one ($\binom{10}{3} = 120$ choices, each with a coefficient of $2^3 = 8$) and the rest of degree zero (1 choice, with a coefficient of 1), for a total contribution of $120 \cdot 8 = 960$. Then $960 + 180 = 1140$ is our answer.

F09A24. **1024**. Note that we can only reach points of the form $(x, 0)$ or $(x, 1)$, where $0 \le x \le 10$ and x is an integer. Also note that any walk is determined by the set of x-coordinates between 0 and 9 at which the walk has a vertical step. (For example, there is only one walk that has vertical steps at x-coordinates $0, 3$ and 4: it starts at $(0, 0)$, goes to $(0, 1)$, then to $(3, 1)$, $(3, 0)$, $(4, 0)$, $(4, 1)$, $(10, 1)$ and finally $(10, 0)$, traveling along a straight line between each consecutive pair of listed points.) Moreover, for every subset of $\{0, 1, 2, \ldots, 8, 9\}$, there is a walk that takes vertical steps at those x-coordinates (and possibly also from $(10, 1)$ to $(10, 0)$). Thus, the number of walks is the same as the number of subsets, and this is $2^{10} = 1024$.

Alternatively, using H to denote a horizontal step and V to denote a vertical step, the problem is equivalent to counting the number of strings of 10 Hs and an even number of Vs so that no two Vs are adjacent. Count the strings with n Vs as follows: each of the n Vs must be placed in one of the 11 spaces determined by the 10 Hs, so there are $\binom{11}{n}$ such strings. Hence the desired number of strings is

$$\binom{11}{0} + \binom{11}{2} + \binom{11}{4} + \ldots + \binom{11}{10} = \frac{1}{2} \sum_{i=0}^{11} \binom{11}{i} = \frac{1}{2} \cdot 2^{11} = 1024.$$

Alternatively, note that there are two types of horizontal step: $(x, 0) \to (x + 1, 0)$ for some x and $(x, 1) \to (x + 1, 1)$ for some x. Denote the first type by H_0 and the second type by H_1. Every path is uniquely determined by a sequence of 10 horizontal steps of these two types, and every sequence of 10 horizontal steps of these two types determines a unique path. Thus the total number of paths is the same as the total number of strings, which is 2^{10} because we have 10 terms and for each term we must independently choose from two choices (H_0 and H_1).

F09A25. **25%** or **25**. Adding up 11%+13%+19% includes every mathematician who has at least two of these properties, but it counts those mathematicians with all three properties three times. Thus, our answer is $11\% + 13\% + 19\% - 3 \cdot 6\% = 25\%$.

F09A26. $\left(\mathbf{\dfrac{5-3\sqrt{5}}{5}}, \mathbf{\dfrac{10-6\sqrt{5}}{5}}\right)$. Rewrite the given equation as $(x-1)^2 + (y-2)^2 = 9$, so the circle has center $(1, 2)$ and radius 3. The point in question lies on the line joining the center of the circle to the origin, and this line has equation $y = 2x$. Substituting this in to the equation of the circle gives $4x^2 - 8x + x^2 - 2x - 4 = 0$ and so $5x^2 - 10x - 4 = 0$, and solving this equation gives $x = \frac{5 \pm 3\sqrt{5}}{5}$. Since the center is in the first quadrant, we want the intersection point with the smaller x-coordinate, and this is the point $\left(\dfrac{5-3\sqrt{5}}{5}, \dfrac{10-6\sqrt{5}}{5}\right)$.

F09A27. $\left(\mathbf{\dfrac{\sqrt{2}}{2}}, \mathbf{\dfrac{\sqrt{2}}{4}}, \mathbf{\dfrac{\sqrt{2}}{6}}\right), \left(\mathbf{\dfrac{\sqrt{2}}{2}}, -\mathbf{\dfrac{\sqrt{2}}{4}}, \mathbf{\dfrac{\sqrt{2}}{6}}\right)$. Subtracting the second equation from the first gives $a^2 = 9c^2$ so $3c = \pm a$. From the third equation, we see that a and c must have the same sign, so $3c = a$ and thus $2a^2 = 1$ or $a = \frac{\sqrt{2}}{2}$ (since $a > 0$). Then $c = \frac{\sqrt{2}}{6}$. Substituting either of these values into either of the first two equations gives $b^2 = \frac{1}{8}$, so $b = \pm\frac{\sqrt{2}}{4}$.

F09A28. $\mathbf{\dfrac{192\sqrt{3}}{25}}$. $\triangle ABC$ is a right triangle. The altitude of $\triangle APQ$ coincides with the altitude from A of triangle $\triangle ABC$, which has length $\frac{6 \cdot 8}{10} = \frac{24}{5}$. In an equilateral triangle of side-length s, the altitude is of length $\frac{s\sqrt{3}}{2}$, so $\triangle APQ$ has edges of length $\frac{48}{5\sqrt{3}} = \frac{16\sqrt{3}}{5}$. It follows that the area of $\triangle APQ$ is $\frac{1}{2} \cdot \frac{24}{5} \cdot \frac{16\sqrt{3}}{5} = \frac{192\sqrt{3}}{25}$.

F09A29. **4**. Suppose the answer to the question is x. Then we are given that $x = 2 + \sqrt{x}$ and so $x - 2 = \sqrt{x}$ or $x^2 - 4x + 4 = x$ and thus $x^2 - 5x + 4 = 0$. Of the two roots $x = 1$ and $x = 4$, $x = 1$ is extraneous (it is two more than its *negative* square root) and so the answer is 4.

F09A30. **9**. Every odd perfect square is one more than a multiple of 8. Thus, a sum of n odd perfect squares must be congruent to n modulo 8. Since $2009 \equiv 1 \pmod 8$, we must have $n \equiv 1 \pmod 8$. Certainly $n = 1$ is not possible, since 2009 is not a perfect square. The next smallest possible value would be $n = 9$, and a little experimentation shows that (for example) $2009 = 43^2 + 11^2 + 5^2 + 3^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2$, so 9 is achievable. (A computer search can be used to show that there are 3689 different ways to write 2009 as a sum of nine odd perfect squares, ignoring the order of the summands.)

One way to find an expression for 2009 as a sum of 9 perfect squares is to begin from $2025 = 45^2 = 9 \cdot 15^2$, which is too large by exactly 16. Replacing $15^2 + 15^2 = 450$ with $17^2 + 11^2 = 410$ decreases the sum by 40, and replacing three copies of $15^2 + 15^2$ by $17^2 + 13^2 = 458$ increases it by 24, so $2009 = 17^2 + 17^2 + 17^2 + 17^2 + 15^2 + 13^2 + 13^2 + 13^2 + 11^2$.