NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Soph-Frosh Division **CONTEST NUMBER 1**

PART I ***FALL 2009*** ***CONTEST 1*** ***TIME: 10 MINUTES***

F09SF1 The average of five numbers is $\frac{1}{3}$ and the average of three of these numbers is $\frac{1}{5}$. Compute the average of the other two numbers.

F09SF2 One leg of a right triangle is length 20. The lengths of the other leg and the hypotenuse are consecutive odd integers. Compute the length of the hypotenuse.

PART II ***FALL 2009*** ***CONTEST 1*** ***TIME: 10 MINUTES***

F09SF3 Compute the two-digit number that is 9 times the sum of its digits.

F09SF4 A boat can travel 8 miles per hour in still water. If it can travel 15 miles with the current in the same time it travels 9 miles against the current, compute the rate of the current in miles per hour.

PART III ***FALL 2009*** ***CONTEST 1*** ***TIME: 10 MINUTES***

F09SF5 There are 4 different science, 3 different math and 2 different history books on a shelf. Compute the number of ways they can be arranged if all the books on the same subject have to be next to each other.

F09SF6 $ABCD$ is a rectangle and P is a point inside the rectangle. If PA=3, PB=4, PC=5, compute PD.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Soph-Frosh Division **CONTEST NUMBER 2**

PART I ***FALL 2009*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F09SF7 Compute the remainder when 777^{777} is divided by 5.

F09SF8 A sphere with radius 5 is cut by a plane which is 3 units from the center of the sphere. Compute the area of the cross-section formed.

PART II ***FALL 2009*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F09SF9 Compute the largest prime factor of 9,991.

F09SF10 If a, b, c, x, y, and z are real numbers such that
$ax + by + cz = 5$
$bx + cy + az = 50$
$cx + ay + bz = 500$
and $a + b + c = 5$, compute $x + y + z$.

PART III ***FALL 2009*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F09SF11 Compute the area of a rectangle whose diagonal is of length 10 and whose length is 5 times its width.

F09SF12 When the numbers 551, 613 and 768 are divided by a, $a > 1$ and an integer, they leave a remainder of b. Compute b.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Soph-Frosh Division **CONTEST NUMBER 3**

PART I ***FALL 2009*** ***CONTEST 3*** ***TIME: 10 MINUTES***

F09SF13 If $\sqrt{x+2}=4$, compute $(x+2)^3$.

F09SF14 Compute the positive value of x such that $x[x]=55$. (Note that $[x]$ denotes the greatest integer that is less than or equal to x. For example, $[1.99]=1$, and $[2]=2$.)

PART II ***FALL 2009*** ***CONTEST 3*** ***TIME: 10 MINUTES***

F09SF15 Compute all ordered pairs (x, y) of integers such that $x + y = xy$.

F09SF16 An old printing press can print a newspaper in 10 hours. A new printing press can print a newspaper in 8 hours. Working together, compute how many hours would it take 3 old and 2 new presses to print a newspaper?

PART III ***FALL 2009*** ***CONTEST 3*** ***TIME: 10 MINUTES***

F09SF17 If 100! is multiplied out, how many zeroes does it end with?

F09SF18 In right triangle ABC, $\angle C$ is a right angle, M is the midpoint of AC, N is the midpoint of BC. If $AN = 12$ and $BM = 14$, compute AB.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Soph-Frosh Division **CONTEST NUMBER 1**

Fall 2009 Solutions

F09SF1 $\mathbf{\dfrac{8}{15}}$**.** The sum of the five numbers is $\dfrac{5}{3}$, the sum of the first three numbers is $\dfrac{3}{5}$, the sum of the other two numbers is $\dfrac{5}{3}-\dfrac{3}{5}=\dfrac{16}{15}$ and their average is $\dfrac{8}{15}$.

F09SF2 **101.** Let x be the length of the other leg and let x + 2 be the length of the hypotenuse. Then $x^2+20^2=(x+2)^2$. Solve to obtain x = 99. Then the length of the hypotenuse is x + 2 = 101. The result is based on the Pythagorean Theorem.

F09SF3 **81.** Let t represent the tens digit and u represent the units digit of the answer. Then $10t + u = 9(t + u)$, so $t = 8u$. Because t and u are digits, t = 8 and u = 1, so the requested number is 81.

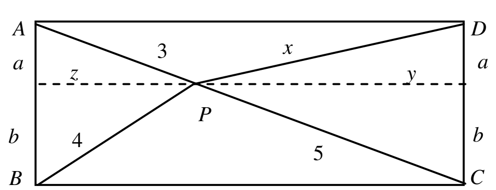
Alternative solution:
We know the answer must be a multiple of 9. Every two-digit multiple of 9 has digit sum equal to 9, so the answer must be $9 \cdot 9 = 81$.

F09SF4 **2 or 2 miles per hour.** Let C be the rate of the current, using the formula $\dfrac{D}{R}=T$, where as usual D = distance, R = rate, and T = time. The boat's rates with and against the current are 8 + C and 8 – C, respectively. Because the times for the two trips are the same, $\dfrac{15}{8+C}=\dfrac{9}{8-C}$. Solve to obtain C = 2.

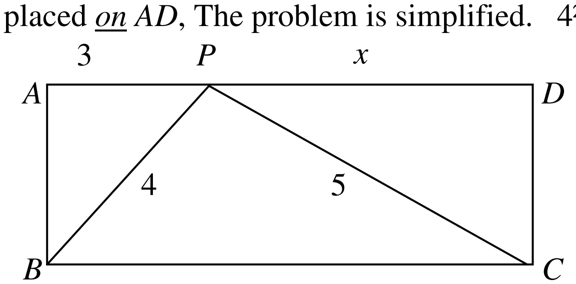
F09SF5 **1728.** There are 3! = 6 ways of arranging the subjects, 4! = 24 ways of arranging the science books, 3! = 6 ways of arranging the math books, and 2! =2 ways of arranging the history books. $6\times24\times6\times2=1728$.

F09SF6 $\mathbf{3\sqrt{2}}$. Draw a line through P parallel to AD.

$$9 - a^2 = z^2 = 16 - b^2$$
$$x^2 - a^2 = y^2 = 25 - b^2$$
$$x^2 - 9 = 9 \qquad x^2 = 18 \qquad x=\sqrt{18} = 3\sqrt{2}$$



Note: If P is placed <u>on</u> AD, The problem is simplified. $4^2 - 3^2= 5^2 - x^2$ $x=3\sqrt{2}$.



Challenge: Show that for any rectangle $ABCD$ and any point P, $PA^2 + PC^2 = PB^2 + PD^2$.

New York City Interscholastic Mathematics League

Soph-Frosh Division CONTEST NUMBER 2

Fall 2009 Solutions

F09SF7 **2.** The unit digit of 777^n goes in cycles of 7, 9, 3, 1, 7, 9, 3, 1 etc. as n increases. Thus the remainder upon division by 5 cycles as 2, 4, 3, 1. Since the exponent 777 is one more than a multiple of 4, the requested remainder is 2.

F09SF8 **16π.** The cross section is a circle with radius 4 since it is one leg of a 3-4-5 right triangle. So its area is 16π.

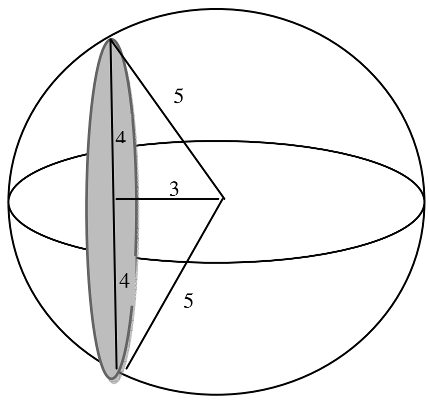


Diagram: The cross section is shaded in gray.

F09SF9 **103.** Since 9991 = 10000 - 9, we may factor it as a difference of squares: $9991 = 100^2 - 3^2 = (100 + 3)(100 - 3) = 103\times97$. Since 103 is prime, it is the largest prime factor of 9991.

F09SF10 **111**. Add the three equations to obtain $(ax + by + cz) + (bx + cy + az) + (cx + ay + bz) = 555$ and then factor to get $(a + b + c)(x + y + z) = 555$. Because $a + b + c = 5$, then $x + y + z = 111$.

F09SF11 $\mathbf{\frac{250}{13}}$ **or** $\mathbf{19\frac{3}{13}}$**.** Let $5x$ be the length and x be the width. Then $(5x)^2 + x^2 = 10^2$, so $26x^2 = 100$. Then the area is $(5x)(x) = 5x^2 = 5(\frac{100}{26}) = \frac{250}{13}$.

F09SF12 **24.**

$$613 = xa + b$$
$$551 = ya + b$$
$$62 = xa - ya = a(x + y)$$

Because the three numbers leave the same remainder when divided by a, the difference between any two of them is a multiple of a. Thus a is a divisor of both 613 – 551 = 62 and 768 – 613 = 155. The only numbers that divides both 62 and 155 are 1 and 31, so $a = 31$. Divide any of the three number by 31 to find that $b = 24$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Soph-Frosh Division **CONTEST NUMBER 3**

Fall 2009 Solutions

F09SF13 **4096.** $\sqrt{x+2}=4$ Square both sides to obtain $x + 2 = 16$. Then $(x + 2)^3 = 16^3 = 4096$.

F09SF14 $\mathbf{\frac{55}{7}}$ **or** $\mathbf{7\frac{6}{7}}$. We have 7 [7] = 49, so 7 is too small. We also have 8 [8] = 64, so 8 is too large. Thus x must be between 7 and 8. We conclude that [x] = 7, so $7x = 55$ and $x=\frac{55}{7}$.

F09SF15 **(0,0) and (2,2).** The given equation is equivalent to $xy - x - y = 0$, thus $xy - x - y + 1 = 1$ so $(x - 1)(y - 1) = 1$. Either $x - 1 = y - 1 = 1$ or $x - 1 = y - 1 = -1$. These give solutions (0,0) and (2,2).

F09SF16 $\mathbf{\frac{20}{11}}$ **or** $\mathbf{1\frac{9}{11}}$. Working x hours, an old press will print $\frac{x}{10}$ of the paper, a new press will print $\frac{x}{8}$ of the paper. Then $3\frac{x}{10}+2\frac{x}{8}=1$, multiply by 40 to get $12x+10x=40$. Then $x=\frac{40}{22}=\frac{20}{11}$.

F09SF17 **24.** You get a zero at the end of a number if you multiply by $10 = 2\times 5$. There are many more factors of 2 than 5 so you have to count how many factors of 5 are there. Every multiple of 5 will therefore produce a zero. There are 20 multiples of 5. Also, 25, 50, 75, and 100 each have two factors of 5. 20 + 4 = 24.

F09SF18 $\mathbf{4\sqrt{17}}$. In ΔACN $x^2 + (2y)^2 = 144$

In ΔMCB $(2x)^2 + y^2 = 196$

Add the two equations to get:

$5x^2 + 5y^2 = 340$

$4x^2 + 4y^2 = AB^2 = 272$

$AB=\sqrt{272}=4\sqrt{17}$.