New York City Interscholastic Math League
Senior Division A Contest Number 1

Part I	Spring 2010	Contest 1	Time: 10 Minutes

S10A1 An arithmetic sequence with twelve terms has third term equal to 6 and seventh term equal to 16. Compute the sum of all twelve terms of this sequence.

S10A2 There are three quadrilaterals whose vertices are $(-2, 6)$, $(-2, -2)$, $(1, -2)$ and $(-1, 0)$, and they have three different areas. Compute the largest of these three areas.

Part II	Spring 2010	Contest 1	Time: 10 Minutes

S10A3 Compute the number of ways in which one can mark three 1×1 squares in a 3×3 square grid so that no two of the marked squares share either an edge or a vertex.

S10A4 In $\triangle ABC$, $AB = BC = 2$ and $AC = 1$. Median $\overline{AM}$ and angle bisector $\overline{CT}$ intersect at point P. Compute the length BP.

Part III	Spring 2010	Contest 1	Time: 10 Minutes

S10A5 Compute the smallest positive integer n such that $2^n - 1$ is divisible by 257.

S10A6 Tanya and Oleg play the following game: from a pile of twelve identical stones, Tanya removes and discards either one, two or three stones. Then Oleg removes either one, two or three of the remaining stones, and then Tanya removes either one, two or three stones, and so on. The game ends when the last stone has been removed. Compute the number of possible games that Tanya and Oleg can play.

New York City Interscholastic Math League
Senior Division A   Contest Number 2

Part I	Spring 2010	Contest 2	Time: 10 Minutes

S10A7 An ant walks one meter at a speed of 15 inches per minute, one meter at a speed of 20 inches per minute and one meter at a speed of 25 inches per minute. Compute (in inches per minute) the average speed of the ant over its entire walk.

S10A8 Compute the number of ordered triples (x, y, z) of positive integers such that $x + y + z \leq 6$.

Part II	Spring 2010	Contest 2	Time: 10 Minutes

S10A9 Compute $2011^6 - 6 \cdot 2011^5 \cdot 2009 + 15 \cdot 2011^4 \cdot 2009^2 - 20 \cdot 2011^3 \cdot 2009^3 + 15 \cdot 2011^2 \cdot 2009^4 - 6 \cdot 2011 \cdot 2009^5 + 2009^6$.

S10A10 A sequence $a_0, a_1, a_2, \ldots$ is defined by $a_0 = 0$, $a_n = a_{n-1} + 2$ if n is odd, and $a_n = 3a_{n-1}$ if n is even. Compute $a_0 + a_1 + a_2 + \ldots + a_{15}$.

Part III	Spring 2010	Contest 2	Time: 10 Minutes

S10A11 Rectangles $ABCD$ and $A'BC'D$ share diagonal BD, as shown. If $AB = A'B = 1$, $BC = BC' = 3$, compute the area of overlap of the two rectangles.
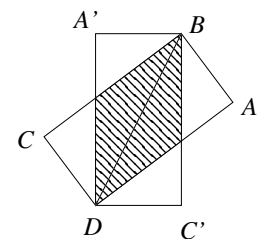
S10A12 Ann randomly selects two unit squares in a 6×6 square grid and marks them. Then, she marks the smallest possible number of additional unit squares so that the result is symmetric under all rotations and reflections of the large grid. Compute the expected number of unit squares that Ann will have marked at the end of this process.

New York City Interscholastic Math League
Senior Division A
Contest Number 3

S10A13   Real numbers x and y are both bigger than 1, and $\log_x y + \log_y x = \dfrac{29}{10}$. Compute $|\log_x y - \log_y x|$.

S10A14   In $\triangle ABC$, we have $AB = 15$, $BC = 9$ and $CA = 12$. Let G be the centroid of $\triangle ABC$. Let line ℓ be the line parallel to $\overline{BC}$ that passes through the midpoint of $\overline{AG}$. Let B' be the intersection of ℓ with $\overline{AB}$ and let C' be the intersection of ℓ with $\overline{AC}$. Compute the area of quadrilateral $BB'C'C$.

S10A15   Compute the number of positive divisors of $75600 = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7$ that are *not* divisors of $25725 = 3 \cdot 5^2 \cdot 7^3$.

S10A16   There is exactly one integer q such that $0 \le q \le 100$ with the property that the polynomial $\dfrac{5n^4}{12} - \dfrac{7n^3}{3} + \dfrac{qn^2}{96} + \dfrac{n}{3} - 3$ takes an integer value whenever n is an integer. Compute q.

S10A17   Compute the volume of the regular octahedron whose vertices are $(10, 0, 0)$, $(0, 10, 0)$, $(0, 0, 10)$, $(-10, 0, 0)$, $(0, -10, 0)$, and $(0, 0, -10)$.

S10A18   Compute the number of 4-tuples (a, b, c, d) of integers such that a, b, c and d lie between 0 and 10, *inclusive*, and $ad - bc$ is not divisible by 11.

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE
SENIOR DIVISION A   CONTEST NUMBER 4

PART I	SPRING 2010	CONTEST 4	TIME: 10 MINUTES

S10A19 Compute the ordered pair (a, b) of real numbers such that we have $(a + bi)(-1 + 2i) = \dfrac{1 + 3i}{2 - 5i}$, where $i = \sqrt{-1}$.

S10A20 Compute the maximum value of the function $\cos\left(x + \frac{\pi}{6}\right) - \cos\left(x - \frac{\pi}{3}\right)$ as x varies over the real numbers.

PART II	SPRING 2010	CONTEST 4	TIME: 10 MINUTES

S10A21 Compute the number of paths that begin at $(0, 0)$, end at $(2, 2)$, and consist of *six* steps such that each step is of unit length and parallel to one of the coordinate axes. (Thus, for example, the first step of the path must be from $(0, 0)$ to one of $(1, 0)$, $(0, 1)$, $(-1, 0)$ or $(0, -1)$.)

S10A22 Compute the number of integers x such that $1 \leq x \leq 105$, $x^2 - 1$ is divisible by 3, x^2 is divisible by 7 and $x^2 + 1$ is divisible by 5.

PART III	SPRING 2010	CONTEST 4	TIME: 10 MINUTES

S10A23 Compute the number of real roots of the equation $(x - 2)^2(6 - x)^2 = 25$.

S10A24 Tanya and Oleg play the following game (called "two-pile Nim"): from two (distinguishable) piles of four indistinguishable stones, the players take turns choosing a pile and removing at least one and at most three stones from that pile. The game ends when the last stone has been removed. Compute the number of possible games in which Tanya moves first.

New York City Interscholastic Math League
## Senior Division A   Contest Number 5

S10A25

Alejandro rolls a fair six-sided die numbered from 1 to 6. Martina rolls a fair ten-sided die with one side numbered 1, two sides numbered 2, three sides numbered 3 and four sides numbered 4. Compute the probability that Alejandro's roll is larger than Martina's roll.

S10A26

Circle O_1 has radius 3 and circle O_2 has radius 1. If the length of the common external tangent of circles O_1 and O_2 is twice the length of their common internal tangent, compute the distance O_1O_2.
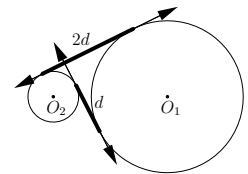
S10A27

Compute the number of ways to write 45 as a sum of two or more consecutive positive integers.

S10A28

Compute the largest real number d such that every value of x that satisfies $|x + 2| \leq d$ also satisfies $|x^2 - 4| \leq \frac{1}{3}$.

S10A29

Compute all ordered pairs (a, b) real numbers such that

$$a^2 + 2ab = 4 = 4ab - b^2.$$

S10A30

A regular octahedron has vertices at $(10, 0, 0)$, $(0, 10, 0)$, $(0, 0, 10)$, $(-10, 0, 0)$, $(0, -10, 0)$, and $(0, 0, -10)$. Compute the number of points (x, y, z) with integer coordinates that are either interior or boundary points of this octahedron. (This includes vertices, points on the edges, points on the faces, and points in the interior of the octahedron.)

S10A1. **177**. Let the first term be a and the common difference be d. Then $a + 2d = 6$ and $a + 6d = 16$. Subtracting one equation from the other yields $d = \frac{5}{2}$ and so $a = 1$. Thus the sum of the twelve terms of the sequence is $12 \cdot \frac{2a+11d}{2} = 177$.

S10A2. **9**. For a polygon R, let $[R]$ denote its area. Label the points $A = (-2, 6)$, $B = (-2, -2)$, $C = (1, -2)$ and $D = (-1, 0)$. Then $\triangle ABC$ is a right triangle with D in its interior, and the three quadrilaterals are $ABCD$, $ABDC$ and $ADBC$. We have $[ABCD] = [ABC] - [CDA]$, $[ABDC] = [ABC] - [BDC]$ and $[ADBC] = [ABC] - [ADB]$. We can use the triangle area formula to compute $[ABC] = \frac{1}{2} \cdot 8 \cdot 3 = 12$, $[BDC] = \frac{1}{2} \cdot 2 \cdot 3 = 3$, $[ADB] = \frac{1}{2} \cdot 8 \cdot 1 = 4$ and so $[CDA] = [ABC] - [ADB] - [BDC] = 5$. It follows that $[ABCD] = 7$, $[ABDC] = 9$ and $[ADBC] = 8$, so the answer is 9.

S10A3. **8**. There are many ways of demonstrating that the only possible arrangements are the two shown at right and their rotations.
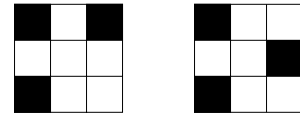
S10A4. $\dfrac{\sqrt{34}}{4}$. Since M is the midpoint of $\overline{BC}$ we have $CM = 1$ and thus $\triangle AMC$ is isosceles. It follows that the angle bisector $\vec{CT}$ is also a median of $\triangle AMC$, so P is the midpoint of $\overline{AM}$. By Stewart's theorem, we have that the length m of the median to the side of length c of a triangle with sides of length a, b and c is given by $m = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$. Thus, $AM = \frac{1}{2}\sqrt{8 + 2 - 4} = \frac{\sqrt{6}}{2}$ and so $BP = \frac{1}{2}\sqrt{8 + 2 - \frac{3}{2}} = \frac{\sqrt{34}}{4}$.

S10A5. **16**. We have $257 = 256 + 1 = 2^8 + 1$. Thus, 257 divides $2^{16} - 1 = (2^8 - 1)(2^8 + 1)$. There are several different ways to see that 257 does not divide $2^n - 1$ for any $n \leq 15$: for example, if 257 divides $2^n - 1$ then it also divides $2^{n-8}(257) - (2^n - 1) = 2^{n-8} + 1$, and $2^{n-8} + 1 < 257$ if $n < 16$. Alternatively, one could design an argument around the observation that $2^8 \equiv -1 \pmod{257}$.

S10A6. **927**. Let a_n be the number of games that can be played if we begin with n stones. Thus, for example, $a_1 = 1$, $a_2 = 2$ and $a_3 = 4$. For $n \geq 4$, the first player may leave $n - 1$, $n - 2$ or $n - 3$ stones. It follows immediately that $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, and so we may recursively compute that $a_4 = 4 + 2 + 1 = 7$, $a_5 = 7 + 4 + 2 = 13$, etc., leading to $a_{12} = 927$. (The numbers in this sequence are sometimes referred to as the "tribonacci numbers.") Follow-up question: if the goal is to take the last stone, which player has a winning strategy in this game?

S10A7. $\dfrac{\mathbf{900}}{\mathbf{47}}$. Suppose that 1 meter is equal to d inches. Then the total time required by the ant is $\frac{d}{15} + \frac{d}{20} + \frac{d}{25}$ minutes, and in this time it travels $3d$ inches. Thus, its average speed is

$$\frac{3}{\frac{1}{15} + \frac{1}{20} + \frac{1}{25}} = \frac{900}{47}$$

inches per minute.

S10A8. **20**. One possible approach is to note that if (x, y, z) is a positive-integer solution of the given inequality and $w = 7 - x - y - z$ then w is a positive integer and $w + x + y + z = 7$. This is a one-to-one correspondance between triples satisfying the given inequality and four-tuples satisfying this new equation. Now we may use the method of "stars and bars" to see that there are

$$\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3!} = 20$$

solutions.

One of several alternative approaches is to solve separately the equations $x + y + z = 3$, $x + y + z = 4$, etc., and add up the number of solutions in each case. Challenge: can you see how to use this idea to provide a general proof of the "hockey-stick identity" by showing that the two sides are different ways of counting the same thing?

S10A9. **64**. The given expression is the binomial theorem expansion of $(2011 - 2009)^6 = 2^6 = 64$.

S10A10. **19648**. We can compute that $a_1 = 2$, $a_2 = 6$, $a_3 = 8$, $a_4 = 24$ and $a_5 = 26$. These values lead to the guess $a_{2n-1} = 3^n - 1$ and $a_{2n} = 3^{n+1} - 3$, which we can prove by induction. Then we have that

$$(a_0 + a_2 + \ldots + a_{14}) + (a_1 + a_3 + \ldots + a_{15}) = (3^1 + \ldots + 3^8 - 8 \cdot 3) + (3^1 + \ldots + 3^8 - 8 \cdot 1)$$

$$= 2 \cdot \frac{3^9 - 3}{2} - 32$$

$$= 3^9 - 35$$

$$= 19648.$$

Alternatively, rather than cleverly guessing the formula, we can examine the base-3 representations of the numbers a_n: we have $a_0 = 0$, we get a_{2n} from a_{2n-1} by adding a 0 at the end of the base-3 expansion, and we get a_{2n+1} from a_{2n} by chaning the final 0 in the base-3 expansion to a 2, so $a_{2n} = 22 \cdots 20_3$ and $a_{2n+1} = 22 \cdots 22_3$, giving us the expressions above.

S10A11. $\dfrac{\mathbf{5}}{\mathbf{3}}$. Let P be the intersection of $\overline{AD}$ and $\overline{BC'}$ and let $AP = x$. By symmetry, $C'P = x$ and so $BP = 3 - x$. Then by the Pythagorean Theorem in $\triangle ABP$ we have that $x^2 + 1 = (3 - x)^2$ and so $x = \frac{4}{3}$. Thus the area of $\triangle ABP$ is $\frac{1}{2} \cdot \frac{4}{3} \cdot 1 = \frac{2}{3}$, and so the area of the region in question is $3 - 2 \cdot \frac{2}{3} = \frac{5}{3}$.

S10A12. $\dfrac{\mathbf{1276}}{\mathbf{105}}$. Consider the diagram below. For any $i \in \{1, \ldots, 6\}$, if Ann marks a square labelled i then she must mark all other squares labelled i as well. Thus, we have five cases:

- Ann marks two cells with label 1 (or equivalently with label 2, or 3). The probability that this happens is $\frac{4}{36} \cdot \frac{3}{35}$ and it results in 4 marked squares. Thus, the total contribution in this case is $\frac{3 \cdot 4 \cdot 4 \cdot 3}{36 \cdot 35}$.

1	4	5	5	4	1
4	2	6	6	2	4
5	6	3	3	6	5
5	6	3	3	6	5
4	2	6	6	2	4
1	4	5	5	4	1

- Ann marks two cells with label 4 (or equivalently with label 5, or 6). The probability that this happens is $\frac{8}{36} \cdot \frac{7}{35}$ and it results in 8 marked squares. Thus, the total contribution in this case is $\frac{3 \cdot 8 \cdot 8 \cdot 7}{36 \cdot 35}$.

- Ann marks one cell with label 1 and one cell with label 2 (or equivalently 1 and 3, or 2 and 3). The probability that this happens is $\frac{8}{36} \cdot \frac{4}{35}$ and it results in 8 marked squares. Thus, the total contribution in this case is $\frac{3 \cdot 8 \cdot 8 \cdot 4}{36 \cdot 35}$.

- Ann marks one cell with label 4 and one cell with label 5 (or equivalently 4 and 6, or 5 and 6). The probability that this happens is $\frac{16}{36} \cdot \frac{8}{35}$ and it results in 16 marked squares. Thus, the total contribution in this case is $\frac{3 \cdot 16 \cdot 16 \cdot 8}{36 \cdot 35}$.

- Ann marks one cell with label 1 and one cell with label 4 (or equivalently 1 and 5, 1 and 6, 2 and 4, 2 and 5, 2 and 6, 3 and 4, 3 and 5, or 3 and 6). The probability that this happens is $\frac{8}{36} \cdot \frac{4}{35} + \frac{4}{36} \cdot \frac{8}{35}$ and it results in 12 marked squares. Thus, the total contribution in this case is $\frac{9 \cdot 12 \cdot (8 \cdot 4 + 4 \cdot 8)}{36 \cdot 35}$.

Finally, we sum up these individual contributions to get the answer $\frac{1276}{105}$.

S10A13. $\dfrac{21}{10}$. Let $\ell = \log_x y$. By logarithm rules, $\frac{1}{\ell} = \log_y x$. Thus $\ell + \frac{1}{\ell} = \frac{29}{10}$, and we can solve this to find that $\ell = \frac{5}{2}$ or $\frac{2}{5}$. In either case, we have $\left|\log_x y - \log_y x\right| = \left|\frac{5}{2} - \frac{2}{5}\right| = \frac{21}{10}$.

S10A14. **48**. By the choice of ℓ and the points B', C', we have that $\triangle ABC \sim \triangle AB'C'$. Let M and M' be the midpoints of BC and $B'C'$, respectively. Since the centroid of a triangle trisects its medians we have $AG = \frac{2AM}{3}$, while by the definition of midpoint we have $AM' = \frac{AG}{2} = \frac{AM}{3}$. This implies immediately that the ratio of similarity between $\triangle ABC$ and $\triangle AB'C'$ is $\frac{1}{3}$, and so the smaller triangle has area exactly $\left(\frac{1}{3}\right)^2 [\triangle ABC] = \frac{1}{9} \cdot 54 = 6$. The area of $BB'C'C$ is simply the difference between the areas of the two triangles, or $54 - 6 = 48$.

Observe that we didn't use the fact that $\triangle ABC$ at all (and in fact our argument works for any triangle). Because the triangle is nice, we could also solve this problem using a coordinate-based approach (or possibly several other methods).

S10A15. **108**. Since $75600 = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7$, there are $(4+1)(3+1)(2+1)(1+1) = 120$ divisors of 75600. A number divides 75600 and 25725 if and only if it divides their GCD, $3 \cdot 5^2 \cdot 7$. The number of such numbers is $(1+1)(2+1)(1+1) = 12$. Thus the answer is $120 - 12 = 108$.

S10A16. **56**. Let $P(n) = \frac{5n^4}{12} - \frac{7n^3}{3} + \frac{qn^2}{96} + \frac{n}{3} - 3$. We know that $\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{4!} = \frac{n^4 - 6n^3 + 11n^2 - 6n}{24}$ is an integer for every integer n. Thus, $Q(n) = P(n) - 10\binom{n}{4} = \frac{n^3}{6} + \frac{(q-440)n^2}{96} + \frac{17n}{6} - 3$ is an integer for every integer n. Similarly, $\frac{(n+1)n(n-1)}{6} = \frac{n^3 - n}{6}$ is an integer for every integer n, so $Q(n) - \frac{n^3 - n}{6} = \frac{(q-440)n^2}{96} + 3n - 3$ is an integer for every integer n, and thus also $\frac{(q-440)n^2}{96}$ must be an integer for every integer n. But then $\frac{q-440}{96}$ must be an integer. The only integer in the range $\left[\frac{-440}{96}, \frac{-340}{96}\right]$ is -4, which means $q - 440 = -4 \cdot 96$ and so $q = 56$.

Alternatively, one can plug in $n = 1$ to see that we must have that $\frac{5}{12} - \frac{7}{3} + \frac{q}{96} + \frac{1}{3} - 3 = \frac{q-440}{96}$ is an integer, and conclude in the same way. (Note that this second solution shows that $q = 56$ is the only possible value, but doesn't prove that when $q = 56$ we have that the value of the polynomial is an integer whenever n is an integer.)

S10A17. $\frac{4000}{3}$. The octahedron consists of two square pyramids glued together along their base. Each of the two square pyramids has volume $\frac{1}{3} \cdot 10 \cdot (10\sqrt{2})^2 = \frac{2000}{3}$, so the total volume is $\frac{4000}{3}$.

S10A18. **13200**. First choose a and b arbitrarily so that we do not have $a = b = 0$. There are $11^2 - 1 = 120$ ways to make this choice. Now, consider the possibilities for choosing c and d. If $a \neq 0$ then there is some integer x such that $ax \equiv c \pmod{11}$. Then as long as $d \not\equiv bx \pmod{11}$, we will have $ad - bc \not\equiv 0 \pmod{11}$. This means that for every one of the 11 choices for c, there are exactly 10 good choices for d, so 110 choices for the pair (c, d). If instead $a = 0$ then $b \neq 0$ and so we may repeat the same argument with the roles of c and d switched, so there are 110 choices for (c, d) in this case, as well. Thus in total we have $120 \cdot 110 = 13200$ choices of (a, b, c, d).

Challenge: what is the answer if we replace "11" with an arbitrary prime p? What step goes wrong if we use a non-prime?

S10A19. $\left(\frac{7}{29}, \frac{3}{29}\right)$. We have $1 + 3i = (a + bi)(-1 + 2i)(2 - 5i) = (a + bi)(8 + 9i) = (8a - 9b) + (9a + 8b)i$. Therefore $8a - 9b = 1$ and $9a + 8b = 3$. Solving this system gives $a = \frac{7}{29}$ and $b = \frac{3}{29}$. Alternatively, expand the left side and multiply the right side by $\frac{2+5i}{2+5i}$ to get $(-a - 2b) + (2a - b)i = \frac{1}{29}(-13 + 11i)$, equate real and imaginary parts, and solve. Or, divide both sides by $-1 + 2i$ and simplify the right-hand side to avoid solving any linear equations.

S10A20. $\sqrt{2}$. Set $y = x - \frac{\pi}{12}$. Then we're trying to maximize $\cos\left(y + \frac{\pi}{4}\right) - \cos\left(y - \frac{\pi}{4}\right)$. Expanding this out using the formulas for the cosine of a sum and difference gives $\cos\left(y + \frac{\pi}{4}\right) - \cos\left(y - \frac{\pi}{4}\right) = -\sqrt{2}\sin y$, so its maximum value is $\sqrt{2}$ (achieved whenever $\sin y = -1$).

S10A21. **120**. There are two possible ways a path of length six can go from $(0,0)$ to $(2,2)$: it may consist either of three steps up, two steps right and one step down or of two steps up, three steps right and one step left. In either case, every path comes from permuting the six steps in some order. In the first case, there are $\frac{6!}{3!2!1!} = 60$ possible orders, and in the second case there are $\frac{6!}{2!3!1!} = 60$ possible orders, so in total we have $60 + 60 = 120$ paths of the desired sort.

S10A22. **4**. We have $x^2 - 1$ is divisible by 3 if and only if x is not divisible by 3, i.e., if and only if $x \equiv 1$ or $x \equiv 2 \pmod 3$. We have x^2 is divisible by 7 if and only if $x \equiv 0 \pmod 7$. Finally, we have $x^2 + 1$ is divisible by 5 if and only if $x \equiv 2$ or $x \equiv 3 \pmod 5$. This gives us four systems of modular equations to solve: we choose one of the two congruences for x modulo 3 and one of the two congruences for x modulo 5, and the congruence $x \equiv 0 \pmod 7$. By the Chinese Remainder Theorem, each of these systems leads to a unique solution modulo $3 \cdot 5 \cdot 7 = 105$, and thus to a unique integer solution in the range $[1, 105]$. These four solutions must be distinct, since they differ in some mod. Thus, our answer is 4. (The actual solutions are 7, 28, 77 and 98.)

Alternatively, one could just check all 10 multiples of 7 that are not also multiples of 3 to see which ones work.

S10A23. **2**. Subtract 25 from both sides and factor as a difference of squares to get $((x - 2)(x - 6) - 5)((x - 2)(x - 6) + 5) = 0$. Thus either $x^2 - 8x + 7 = 0$, whose roots 1 and 7 are both real, or $x^2 - 8x + 17 = 0$, whose roots are non-real (the discriminant is $8^2 - 4 \cdot 17 = -4 < 0$). Thus there are exactly two real roots.

S10A24. **784**. Let $a(m, n)$ be the number of games that can be played beginning with m stones in one pile and n in the other. Then we have that $a(0, 0) = 1$, $a(m, n) = 0$ if either of m or n is not a nonnegative integer, and in general $a(m, n) = a(m - 1, n) + a(m - 2, n) + a(m - 3, n) + a(m, n - 1) + a(m, n - 2) + a(m, n - 3)$. We can then use this to compute relatively swiftly the total number of games. It may help to place the values of $a(m, n)$ in a table for easy computation, as shown.

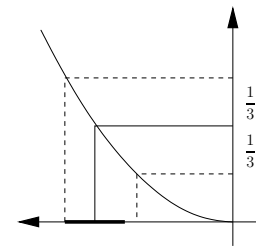| $a(m,n)$	0	1	2	3	4
0	1	1	2	4	7
1	1	2	5	12	26
2	2	5	14	37	89
3	4	12	37	106	277
4	7	26	89	277	784

S10A25. $\frac{1}{2}$. If Alejandro rolls a 1, the probability is 0. If he rolls a 2, the probability is $\frac{1}{10}$. If he rolls a 3, the probability is $\frac{3}{10}$. If he rolls a 4, the probability is $\frac{6}{10}$, and if he rolls 5 or 6, the probability is 1 each. Thus, the overall probability is $\frac{1}{6} \cdot \left(0 + \frac{1}{10} + \frac{3}{10} + \frac{6}{10} + 1 + 1\right) = \frac{1}{2}$.

S10A26. $\mathbf{2\sqrt{5}}$. Let the length of the internal tangent be d (so the length of the external tangent is $2d$) and let the distance in question be x. Then we have $x^2 = (3-1)^2 + (2d)^2$ and $x^2 = (3+1)^2 + d^2$. Thus $3x^2 = 4(16 + d^2) - (4 + 4d^2) = 60$ and so $d = 2\sqrt{5}$. Challenge: can you show that the two tangent lines in the diagram accompanying the problem (not this solution) are perpendicular?
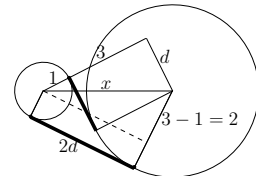
S10A27. $\mathbf{5}$. If we write 45 as a sum of $n + 1$ consecutive positive integers we have $a + (a+1) + \ldots + (a+n) = 45$, so $(n+1)a + \frac{n(n+1)}{2} = 45$ or $(n+1)(2a+n) = 90$. Both a and n must be at least 1, so $n + 2a > n + 1$ and we can choose $n + 1$ to be the smaller of any pair of factors of 90. This leads to $n = 1, 2, 4, 5$ or 8, associated with $a = 22, 14, 7, 5$ or 1, respectively, for a total of five expressions.

S10A28. $\mathbf{-2 + \frac{\sqrt{39}}{3}}$. We have that $|x + 2| \leq d$ if and only if $-2 - d \leq x \leq -2 + d$. Also $|x^2 - 4| \leq \frac{1}{3}$ if and only if $\frac{11}{3} \leq x^2 \leq \frac{13}{3}$. Equivalently, $|x^2 - 4| \leq \frac{1}{3}$ holds if and only if $\frac{\sqrt{33}}{3} \leq x \leq \frac{\sqrt{39}}{3}$ or $-\frac{\sqrt{39}}{3} \leq x \leq -\frac{\sqrt{33}}{3}$ does. Thus, we need to choose the largest value of d such that $[-2 - d, -2 + d] \subseteq \left[-\frac{\sqrt{39}}{3}, -\frac{\sqrt{33}}{3}\right] \cup \left[\frac{\sqrt{33}}{3}, \frac{\sqrt{39}}{3}\right]$. Since the interval $[-2 - d, -2 + d]$ contains a point in the negative part of this union (specifically, -2), the entire interval must be contained there. Thus, we want the largest value of d such that $-2 - d \geq -\frac{\sqrt{39}}{3}$ and $-2 + d \leq -\frac{\sqrt{33}}{3}$, i.e., the largest value d such that $d \leq 2 - \frac{\sqrt{33}}{3}$ and $d \leq -2 + \frac{\sqrt{39}}{3}$. This value is exactly $\min\left(2 - \frac{\sqrt{33}}{3}, -2 + \frac{\sqrt{39}}{3}\right)$. The function $f(x) = \sqrt{x}$ is concave-down, so $-2 + \frac{\sqrt{39}}{3} < 2 - \frac{\sqrt{33}}{3}$ and thus the answer is $-2 + \frac{\sqrt{39}}{3}$.

Alternatively, rather than using the concavity of the square root function, note that the comparison between $2 - \frac{\sqrt{33}}{3}$ and $-2 + \frac{\sqrt{39}}{3}$ is the same as the comparison between $4\sqrt{3}$ and $\sqrt{11} + \sqrt{13}$, which is the same as the comparison between 48 and $11 + 2\sqrt{143} + 13$, which is the same as the comparison between 12 and $\sqrt{143}$, so the latter is smaller than the former.

S10A29. $\left(\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right), \left(-\frac{2\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3}\right)$. From the equality of the left-most and right-most terms, we have $a^2 - 2ab + b^2 = 0$, so $(a - b)^2 = 0$ and thus $a = b$. Then from the equality of the outer terms with the middle term we have $3a^2 = 4$ so $a = \pm\frac{2\sqrt{3}}{3}$.

S10A30. $\mathbf{1561}$. We divide the octahedron into several pieces and count the points in each piece separately. First, we consider how many of the coordinates are equal to 0. If all three coordinates are equal to 0, we have the unique point $(0, 0, 0)$. If two of the coordinates are equal to zero, the remaining coordinate must be some nonzero value between -10 and 10, inclusive. This gives 60 total points (twenty of the form $(a, 0, 0)$, twenty of the form $(0, b, 0)$

and twenty of the form $(0, 0, c)$.) Now suppose that exactly one of the coordinates is equal to zero, say the z-coordinate. A point $(a, b, 0)$ is on or inside the octahedron if and only if all four points $(\pm a, \pm b, 0)$ are inside the octahedron. So, consider the case that $a, b > 0$. The set of points in this quadrant that are also in the octahedron are exactly those points such that $a + b \leq 10$. One can count (for example, by systematic listing or by stars and bars) that there are exactly $\binom{10}{2} = 45$ pairs of positive integers that satisfy this condition. Now we also have to take into account the other possible signs for the coordinates, as well as the cases in which the x- or y-coordinate is 0; this gives a total of $3 \cdot 4 \cdot 45 = 540$ points. Finally, we have to consider the case in which all coordinates are nonzero. We again have that (a, b, c) is in the octahedron if and only if all eight points $(\pm a, \pm b, \pm c)$ are in the octahedron. So, let us first consider the points with $a, b, c > 0$. We need to count the number of such points that lie on or below the plane passing through $(10, 0, 0)$, $(0, 10, 0)$ and $(0, 0, 10)$. This plane has equation $x + y + z = 10$. Thus, we need to count the number of positive-integer solutions to the inequality $a + b + c \leq 10$. Recall from Problem S10A8 that the number of solutions of this inequality is precisely $\binom{10}{3} = 120$. Accounting for signs, the total number of points in this case is $8 \cdot 120 = 960$. Thus in total we have $1 + 60 + 540 + 960 = 1561$ points on or inside the octahedron.

Challenge: let $O(n)$ be the number of points on or inside the regular octahedron with vertices $(\pm n, 0, 0)$, $(0, \pm n, 0)$, $(0, 0, \pm n)$ and let $I(n)$ be the number of points inside (but not on the faces, edges or vertices) of the same octahedron. What is the relationship between these two polynomials? (There is a general phenomenon here which can be considered the three-dimensional generalization of Pick's Theorem. The polynomial $O(n)$ is called the *Ehrhart polynomial* of the octahedron.)