New York City Interscholastic Math League
Junior Division   Contest Number 1

Part I   Spring 2012   Contest 1   Time: 10 Minutes

S12J1 Compute the smallest positive integer q such that there exists a positive integer p satisfying $\frac{1}{5} < \frac{p}{q} < \frac{1}{4}$.

S12J2 Compute the minimum value of n such that a regular polygon with n sides has 100° rotational symmetry.

Part II   Spring 2012   Contest 1   Time: 10 Minutes

S12J3 Compute the number of sets $\{a, b, c\}$ of primes which satisfy $a > b > c$ and $a + b + c = 60$.

S12J4 Line l has equation $3x + 4y = 12$. Line m is one unit from line l and closer to the origin. An equation of line m can be written in the form $ax + by = c$, where a, b, and c are positive integers whose greatest common divisor is 1. Compute the ordered triple (a, b, c).

Part III   Spring 2012   Contest 1   Time: 10 Minutes

S12J5 In triangle ABC, the measures of angles A and B are 58° and 68° respectively, and O is the circumcenter (center of the circumscribed circle). Find the measure of angle OBC in degrees.

S12J6 Three couples are sitting in a row. Compute the number of arrangements in which no person is sitting next to his or her partner.

New York City Interscholastic Math League
Junior Division   Contest Number 2

Part I   Spring 2012   Contest 2   Time: 10 Minutes

S12J7 Find the maximum integer n that is less than 2012 for which the sum $1+2+3+\cdots+n$ is odd.

S12J8 Find the positive integer n for which $\frac{1}{n+1} < \sqrt{960} - \sqrt{959} < \frac{1}{n}$.

Part II   Spring 2012   Contest 2   Time: 10 Minutes

S12J9 Points A, B, C, and D are collinear with $AB = 5, BC = 8$, and $CD = 14$. Compute all possible values of AD.

S12J10 If $4x + 5y = 20$, find the minimum value of $\sqrt{x^2 + y^2}$.

Part III   Spring 2012   Contest 2   Time: 10 Minutes

S12J11 Compute the least positive integer n such that $10! \cdot n = m^3$ for some positive integer m.

S12J12 Compute the coefficient of x^9 in the expansion of $(x^3 + x^2 + 1)^8$.

New York City Interscholastic Math League
Junior Division   Contest Number 3

Part I   Spring 2012   Contest 3   Time: 10 Minutes

S12J13 A certain linear translation maps point $A(-1, 3)$ to point $A'(2, 6)$ and maps point $B(-2, 4)$ to point $B'(p, q)$. Compute the ordered pair (p, q).

S12J14 Find the real value of x that satisfies $\log_{10}((x + 100) - x) = \dfrac{x + 100}{x}$.

Part II   Spring 2012   Contest 3   Time: 10 Minutes

S12J15 Compute $\dfrac{15 \cdot 15! + 15!}{13 \cdot 13! + 13!}$.

S12J16 In triangle ABC, the measures of angles A and B are $58°$ and $68°$ respectively, I is the incenter, and $\overleftrightarrow{AI}$ intersects the circumcircle at D, $D \neq A$. Find the measure of angle BID in degrees.

Part III   Spring 2012   Contest 3   Time: 10 Minutes

S12J17 Compute the number of square numbers between 1000 and 9999, for which the hundreds digit equals the units digit and the thousands digit equals the tens digit.

S12J18 In square $ABCD$, $AB = 5$. Point P is not in the plane of the square, $PA = \sqrt{39}$, $PB = 8$, and $PC = 7$. Compute PD^2.

New York City Interscholastic Math League
Junior Division — Contest Number 1 Solutions

S12J1. **Answer : 9**
Solution. The inequality cannot be satisfied if $p = 1$, and if $p > 1$ we have that $\frac{p}{q} \geq \frac{1}{4}$ for $q \leq 8$. Thus the smallest possible value for q is 9, and indeed $\frac{2}{9}$ satisfies the inequality.

S12J2. **Answer : 18**
Solution. The polygon has 100° rotational symmetry if and only if each vertex "lands" on another vertex after a 100° rotation. The degree measure of each arc intercepted by a side of the polygon is $360/n$, so for a vertex to land on another vertex, it must be rotated $(360/n)m°$ for some integer m. Thus $360m/n = 100$. Hence $18m = 5n$, and so the minimum value of n is 18.

S12J3. **Answer : 3**
Solution. Because $a + b + c$ is even, at least one of them must be even. Therefore one of the primes is 2 and the sum of the other two is 58. Check the primes to find that the only three such sets are $\{2, 5, 53\}$, $\{2, 11, 47\}$, and $\{2, 17, 41\}$.

S12J4. **Answer : $(\mathbf{3}, \mathbf{4}, \mathbf{7})$**
Solution. Line l, the x-axis, and the y-axis determine a right triangle whose legs have lengths 3 and 4. The distance from line l to the origin is the length h of the altitude to the hypotenuse of that right triangle. The area of the right triangle is $\frac{1}{2}(3 \cdot 4)$, and because the hypotenuse is 5, it is also $\frac{1}{2}(5h)$. Thus $h = 12/5$, and so the distance from line m to the origin is $(12/5) - 1 = 7/5$. Line m also forms a right triangle with the x- and y-axes. This right triangle is similar to the one formed by line l, and their similarity ratio is $(7/5) : (12/5) = 7 : 12$. Thus the intercepts of line m are $(0, 21/12)$ and $(28/12, 0)$. Because line m is parallel to line l, it has equation $3x + 4y = k$ for some real number k. Substitute the coordinates of either intercept to find that $k = 7$. Thus $(a, b, c) = (3, 4, 7)$.

S12J5. **Answer : 32**
Solution. Angle A is an inscribed angle so $m\angle BOC = 2(58) = 116$. Because OB and OC are radii of the circumcircle, triangle COB is isosceles, and $m\angle OBC = \frac{1}{2}(180 - 116) = 32$.

S12J6. **Answer : 240**
Solution. Let A denote both the person in the first seat and his or her partner, let B denote both the person in the second seat and his or her partner, and let C denote both of the remaining people. Each permissible seating arrangement corresponds to a six-letter string consisting of two As, Bs, and Cs with no two As, Bs, or Cs adjacent. The first three letters in a permissible string must be ABA or ABC. In the former case, the last three letters must be CBC, while in the latter case, the last three letters may be any arrangement of ABC that does not begin with C, of which there are four: ABC, ACB, BAC, and BCA. Notice that there are 3! possible assignments of the three couples to the letters A, B, and C; and there

are 2 possible arrangements for the members of each couple. Thus each of the 5 permissible strings corresponds to $3! \cdot 2^3 = 48$ seating arrangements, so there are $5 \cdot 48 = 240$ permissible seating arrangements.

New York City Interscholastic Math League
Junior Division   Contest Number 2 Solutions

S12J7. **Answer : 2010**
Solution. Recall that $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$. For $n = 2011$, this gives $\frac{1}{2}n(n+1) = 2011 \cdot 1006$, which is even. For $n = 2010$, this gives $\frac{1}{2}n(n+1) = 1005 \cdot 2011$, which is odd. Thus the maximum n is 2010.

S12J8. **Answer : 61**
Solution. The given condition is equivalent to $n+1 > (\sqrt{960}-\sqrt{959})^{-1} > n$, that is, $n+1 > \sqrt{960}+\sqrt{959} > n$. Because $31^2 = 961$, the sum $\sqrt{960}+\sqrt{959}$ is slightly less than 62. Thus the requested value of n is 61.

S12J9. **Answer : $\mathbf{1, 11, 17, 27}$**
Solution. Because $AB = 5$ and $BC = 8$, conclude that $AC = 8-5 = 3$ or $AC = 8+5 = 13$. Then $AD = 14-3, 14+3, 14-13$, or $14+13$, that is, $AD = 11, 17, 1$, or 27.

S12J10. **Answer : $\mathbf{20\sqrt{41}/41}$**
Solution. The expression $\sqrt{x^2+y^2}$ represents the distance from a point (x, y) to the origin. Therefore, the requested minimum value is the distance from the point on the line with equation $4x+5y = 20$ that is closest to the origin. That line, the x-axis, and the y-axis determine a right triangle whose legs have lengths 4 and 5 and whose hypotenuse has length $\sqrt{41}$. The desired distance h is the altitude to the hypotenuse. Because the area of the triangle is equal to both $\frac{1}{2} \cdot 4 \cdot 5$ and $\frac{1}{2} \cdot \sqrt{41} \cdot h$, conclude that $h = 20\sqrt{41}/41$.

S12J11. **Answer : 4410**
Solution. The prime factorization of 10! is $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$. Then $2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot n = m^3$. Because a number is a cube exactly when all of the exponents in its prime factorization are divisible by 3, the requested value of n is $2 \cdot 3^2 \cdot 5 \cdot 7^2 = 4410$.

S12J12. **Answer : 336**
Solution. Each term in the expanded product before simplifying is obtained by selecting either an x^3, an x^2 or a 1 from each of the eight factors. Denote by a, b, and c the number of factors of x^3, x^2, and 1 respectively that are selected. The resulting product is $(x^3)^a(x^2)^b(1)^c = x^{3a+2b}$. The coefficient of x^9 in the simplified expansion must equal the number of selections for which $3a+2b = 9$ and $a+b+c = 8$. Thus $(a, b, c) = (3, 0, 5)$ or $(1, 3, 4)$. The number of selections consisting of 3 factors of x^3 and 0 factors of x^2 is $\binom{8}{3} = 56$, and the number of selections consisting of 1 factor of x^3 and 3 factors of x^2 is $\binom{8}{1}\binom{7}{3} = 280$. Thus the desired coefficient is $56+280 = 336$.

New York City Interscholastic Math League
Junior Division — Contest Number 3 Solutions

S12J13. **Answer** : $(\mathbf{1}, \mathbf{7})$
Solution. The translation moves point A to point A', 3 units to the right and 3 units up. Thus it must do the same to B, so $B'(1, 7)$.

S12J14. **Answer** : **100**
Solution. Notice that $\log_{10}((x+100)-x) = \log_{10} 100 = 2$, and that $(x+100)/x = 1+100/x$. Thus the given equation is equivalent to $2 = 1 + 100/x$. Then $100/x = 1$ and so $x = 100$.

S12J15. **Answer** : **240**
Solution. Factor to obtain

$$\frac{15 \cdot 15! + 15!}{13 \cdot 13! + 13!} = \frac{15!(15+1)}{13!(13+1)} = \frac{16!}{14!} = 16 \cdot 15 = 240$$

S12J16. **Answer** : **63**
Solution. The incenter of a triangle is the intersection of the angle bisectors, so $m\angle IBC = 34°$ and $m\angle CAD = 29°$. Since angle BID is the external angle of triangle AIB, $\angle BID = \angle ABI + \angle BAI = 34° + 29° = 63°$.

S12J17. **Answer** : **0**
Solution. Let n^2 be a number satisfying the given conditions. Denote its 1000s and 10s digits by a and its 100s and 1s digits by b. Then $n^2 = 1000a + 100b + 10a + b = 101(10a + b)$. Because 101 is a prime, 101 must be a divisor of n, but $1000 \le n^2 \le 9999$ so $n < 100$, and there are no numbers that satisfy the given conditions.

S12J18. **Answer** : **24**
Solution. Let P' be the point in the plane of the square such that PP' is perpendicular to the plane of the square. Let $h = PP'$, and denote $P'A$, $P'B$, $P'C$, and $P'D$ by a, b, c, and d, respectively. Then $PA^2 = h^2 + a^2, PB^2 = h^2 + b^2, PC^2 = h^2 + c^2$, and $PD^2 = h^2 + d^2$. Now let p, q, r, and s denote the distances from P' to $\overline{AD}$, $\overline{BC}$, $\overline{AB}$, and $\overline{CD}$ respectively. Then $p^2+r^2 = a^2, r^2+q^2 = b^2, q^2+s^2 = c^2$, and $s^2+p^2 = d^2$. Thus $a^2+c^2 = p^2+q^2+r^2+s^2 = b^2+d^2$. Hence $a^2 + h^2 + c^2 + h^2 = b^2 + h^2 + d^2 + h^2$. Therefore $PA^2 + PC^2 = PB^2 + PD^2$. This result is known as the British-Flag Theorem, which greatly simplifies this problem. Substitute to find that $PD^2 = 24$.