New York City Interscholastic Math League
Soph-Frosh Division   Contest Number 1

Part I   SPRING 2012   Contest 1   Time: 10 Minutes

S12SF1 Tom's test scores to date are 80, 85, and 90. Compute the minimum test score he would need on his fourth test to raise his test average by at least 3 points.

S12SF2 A digital watch displays hours and minutes in the 12-hour time format (AM or PM). At midnight, it displays 12:00 instead of 00:00. Compute the absolute value of the difference between the largest possible sum and the smallest possible sum of the digits in the display.

Part II   SPRING 2012   Contest 1   Time: 10 Minutes

S12SF3 If $b > a$, $a^2+b^2 = 3^2+4^2+5^2$ is satisfied by only one pair of positive integers (a, b). Compute the value of $a + 3b$.

S12SF4 Five cards are labeled as A, and four other cards are labeled as B. These nine cards are shuffled randomly and arranged in a row. Compute the probability that the arrangement reads $ABABABABA$.

Part III   SPRING 2012   Contest 1   Time: 10 Minutes

S12SF5 x, y, and z are positive, and they satisfy the following equations:

$$\begin{aligned} x^2y &= 28 \\ x^2z &= 56 \\ yz &= 98 \end{aligned}$$

Compute $\frac{z}{xy}$.

S12SF6 Triangle ABC has vertices $A = (3, 0)$, $B = (0, 5)$, and C, where C is on the line $5x + 3y = 30$. Compute the area of $\triangle ABC$.

New York City Interscholastic Math League
Soph-Frosh Division Contest Number 2

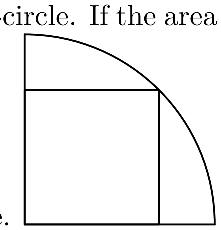
Part I SPRING 2012 Contest 2 Time: 10 Minutes

S12SF7 Compute the sum of all integers, n, that satisfy $n^2 + n^6 = 68$.

S12SF8 Of 100 high school students, two-fifths of them take math and one-fifth of them take art. If 10 students take both subjects, compute the number of students who take neither.

Part II SPRING 2012 Contest 2 Time: 10 Minutes

S12SF9 A square is inscribed in a quarter-circle region as shown below. One vertex of the square lies on an arc of the quarter-circle. If the area of the quarter-circle is 64π, compute the area of the square.



S12SF10 Only one ordered pair of real numbers (x, y) satisfies the following set of equations:

$$\begin{aligned} 5^x - 13y &= 0 \\ 13^x - 5y &= 0 \end{aligned}$$

Compute $x + y$.

Part III SPRING 2012 Contest 2 Time: 10 Minutes

S12SF11 A linear function $y = f(x)$ with positive integer coefficients passes through $(4, 5)$ and $(10, m)$ where $0 < m < 25$. Compute the average of all possible values of m.

S12SF12 A and B each toss one fair coin three times. Compute the probability that they get the same number of tails.

New York City Interscholastic Math League
Soph-Frosh Division   Contest Number 3

Part I   SPRING 2012   Contest 3   Time: 10 Minutes

S12SF13   Compute the positive integer, n, that satisfies the following equation:

$$(3^2 + 5^2) + 13(3^2 + 5^2) + 20(3^2 + 5^2) = n^2$$

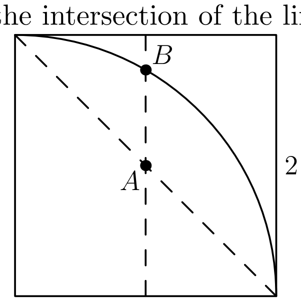
S12SF14   Compute the value of $\frac{x-y}{x+y}$ if

$$\frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{x} - \frac{1}{y}} = 10$$

Part II   SPRING 2012   Contest 3   Time: 10 Minutes

S12SF15   Compute the number of distinct positive integers n where $\sqrt{n}$ differs from $\sqrt{121}$ by less than 2.

S12SF16   In the diagram below, a quarter-circle is inscribed in a square. The length of each side of the square is 2. Point A is the intersection of the diagonal and the line segment connecting midpoints of opposite sides of the square, while point B is the intersection of the line segment and the arc. Compute the length of AB.



Part III   SPRING 2012   Contest 3   Time: 10 Minutes

New York City Interscholastic Math League
Soph-Frosh Division — Contest Number 1 Solutions

S12SF1. $\boxed{\textbf{97.}}$ The sum and the average of the first three test scores are $80+85+90 = 255$ and 85, respectively. To raise his average by at least 3 points, Tom needs:

$$\frac{255+x}{4} = 85+3 = 88 \Longrightarrow x = 97$$

S12SF2. $\boxed{\textbf{22.}}$ First, the smallest sum occurs at $01:00$, and the sum is 1. The largest sum occurs at $09:59$, and its value is 23. Therefore, the absolute value of the difference between these two numbers is 23 - 1 = 22.

S12SF3. $\boxed{\textbf{22.}}$ We have the following equation:

$$a^2 + b^2 = 50$$

Then only one pair satisfies the above equation, $(1,7)$. Since $b > a$, $a = 1$ and $b = 7$. Therefore $a + 3b = 22$.

S12SF4. $\boxed{\textbf{1/126.}}$ The number of distinct arrangements of the nine cards is:

$$\frac{9!}{5!4!} = 126$$

Out of these unique arrangements, there is only one arrangement that reads $ABABABABA$. Therefore, the probability is $\frac{1}{126}$.

S12SF5. $\boxed{\textbf{1.}}$ From the first two equations, we have

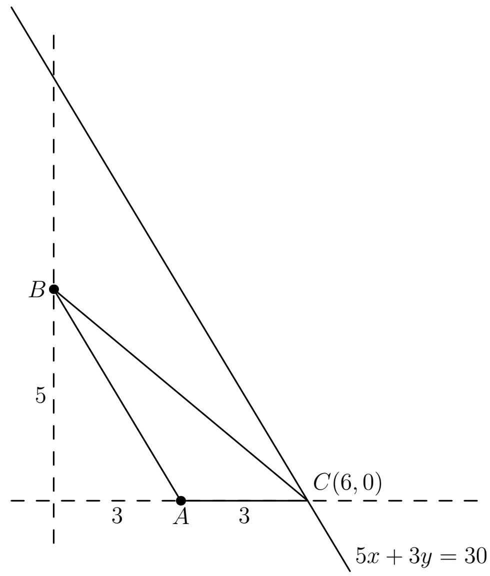
$$\frac{x^2y}{x^2z} = \frac{y}{z} = \frac{1}{2} \Longrightarrow z = 2y$$

Substituting $z = 2y$ into the third equation, we have

$$y(2y) = 2y^2 = 98 \Longrightarrow y = 7$$

Then $z = 2y = 14$ and $x = 2$. Therefore, $\frac{z}{xy} = 1$.

S12SF6. $\boxed{\textbf{7.5 or 15/2.}}$ We can see that the slope of the line $5x + 3y = 30$ is equal to the slope of the line connecting A and B as shown in the figure below. Hence, the two lines are parallel.



Therefore, the area of $\triangle ABC$ does not depend on where C is located on the line. In other words, the area of $\triangle ABC$ will stay the same as long as C is on the given line. To make the problem easy, we can assume that C is on the intersection between the given line and the x-axis, $(6, 0)$.

Then the area of $\triangle ABC$ is

$$\text{Area of } \triangle ABC = \frac{5 \cdot 3}{2} = 7.5$$

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Soph-Frosh Division — Contest Number 2 Solutions

S12SF7. $\boxed{\textbf{0.}}$ Only -2 and 2 satisfy the given equation. Therefore, the sum is $(-2)+2 = 0$.

S12SF8. $\boxed{\textbf{50.}}$ 40 students are in math, and 20 students are in art. Then the number of students who are in either math or art is

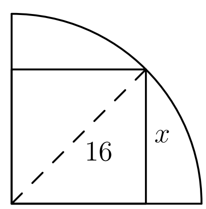
$$40 + 20 - 10 = 50$$

Therefore, 100 - 50 = 50 students are not taking neither.

S12SF9. $\boxed{\textbf{128.}}$ Since the area of a full-circle with radius of r is πr^2, the area of a quarter-circle is $\frac{\pi r^2}{4}$:

$$\frac{\pi r^2}{4} = 64\pi \Longrightarrow r = 16$$

Since the radius of the quarter-circle is 16, the diagonal of the inscribed square is 16 as well: Then by Pythagorean Theorem, we have



$$x^2 + x^2 = 16^2 \Longrightarrow x^2 = 128$$

Since the problem is asking for the area of the square, there is no need to solve for x.

S12SF10. $\boxed{\textbf{-64/65.}}$ We can solve for y in both equations:

$$y = \frac{5^x}{13}$$
$$y = \frac{13^x}{5}$$

$$\frac{5^x}{13} = \frac{13^x}{5} \Longrightarrow 5^{x+1} = 13^{x+1}$$

The above equation is true if and only if $x = -1$. Therefore, the ordered pair is $\left(-1, \frac{1}{65}\right)$, and the sum is $-\frac{64}{65}$.

S12SF11. $\boxed{\textbf{17.}}$ Since $f(x)$ is a linear function that passes through those two points. Then the slope of the line is $\frac{m-5}{6}$. Since the function has positive coefficients, the slope must be a positive integer. There are only three possible values of $0 < m < 25$ that satisfies this condition: $m = 11, 17, 23$. The average of these three numbers is 17.

S12SF12. $\boxed{\textbf{5/16.}}$ We can divide the problem into four cases:

1. They both get no tails
$$\frac{1}{8} \cdot \frac{1}{8} = \frac{1}{64}$$
2. They both get only one tail
$$\binom{3}{1} \cdot \frac{1}{8} \cdot \binom{3}{1} \cdot \frac{1}{8} = \frac{9}{64}$$
3. They both get two tails
$$\binom{3}{2} \cdot \frac{1}{8} \cdot \binom{3}{2} \cdot \frac{1}{8} = \frac{9}{64}$$
4. They both get three tails
$$\frac{1}{8} \cdot \frac{1}{8} = \frac{1}{64}$$

To get the final answer, we just add all the cases:
$$\frac{1}{64} + \frac{9}{64} + \frac{9}{64} + \frac{1}{64} = \frac{5}{16}$$

S12SF17 Compute the y-coordinate of the point on the y-axis which is equidistant from (3,-2) and (5,8).

S12SF18 A solution to the following equation can be expressed as $a+\sqrt{\sqrt{b}}$ where a and b are positive integers. Compute $a+b$.

$$\frac{x^2-6x+10}{x^2-6x+11}+\frac{x^2-6x+12}{x^2-6x+9}=3$$

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S12SF13. $\boxed{\textbf{34.}}$ We can factor out (3^2+5^2) in the given equation:

$$(3^2+5^2)(1+13+20)=34^2$$

Therefore, $n=34$.

S12SF14. $\boxed{\textbf{1/10.}}$ Simplifying the given equation, we have

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}=\frac{x+y}{x-y}=10$$

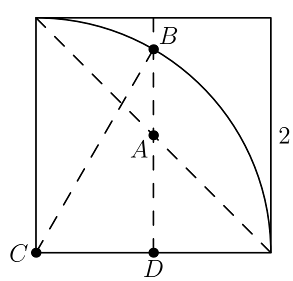
Therefore, $\frac{x-y}{x+y}=\frac{1}{10}$.

S12SF15. $\boxed{\textbf{89.}}$ Since $\sqrt{n}$ differs from $\sqrt{121}=11$ by less than 2, we can set up an inequality:

$$|\sqrt{n}-11|<2 \Longrightarrow -2<\sqrt{n}-11<2 \Longrightarrow 81<n<169$$

Therefore, there are $169-81+1=89$ distinct positive integers.

S12SF16. $\boxed{\sqrt{3}-1.}$ Since the side length of the square is 2, the radius of the quarter-circle is 2 as well. Then we can connect the left bottom corner of the square and point B to form a triangle as shown below: CB is the radius of the circle, 2, and $CD=1$. Using



the Pythagorean Theorem, we can find BD: $BD=\sqrt{2^2-1^2}=\sqrt{3}$. We also know that $AD=1$, and this gives us:

$$AB=BD-AD=\sqrt{3}-1$$

S12SF17. $\boxed{\mathbf{19/5.}}$ Let the coordinate of the point on the y-axis be $(a, 0)$. Then using the distant formula, we have

$$\sqrt{25 + (8 - a)^2} = \sqrt{9 + (a + 2)^2} \Longrightarrow (a + 2)^2 - (8 - a)^2 = 16$$

Using $a^2 - b^2 = (a + b)(a - b)$ factoring, we can simplify the above equation

$$(a + 2 + 8 - a)(a + 2 + a - 8) = 10(2a - 6) = 16 \Longrightarrow a = \frac{19}{5}$$

S12SF18. $\boxed{\mathbf{9.}}$ Since $x^2 - 6x + 9 = (x - 3)^2$, we can define $y = (x - 3)^2$. Rewriting the given equation using y, we get:

$$\frac{y + 1}{y + 2} + \frac{y + 3}{y} = \frac{2y^2 + 6y + 6}{y^2 + 2y} = 3$$

$$3y^2 + 6y = 2y^2 + 6y + 6 \Longrightarrow y^2 = 6 \Longrightarrow y = \sqrt{6}$$

Going back to our definition of y, we have

$$y = (x - 3)^2 = \sqrt{6} \Longrightarrow x = 3 + \sqrt{\sqrt{6}}$$

Therefore, $a = 3$ and $b = 6$ resulting in $a + b = 9$.